Generalized empty-interval method applied to a class of one-dimensional stochastic models

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In this work we study, on a finite and periodic lattice, a class of one-dimensional (bimolecular and single-species) reaction-diffusion models that cannot be mapped onto free-fermion models. We extend the conventional empty-interval method, also called interparticle distribution function (IPDF) method, by introducing a string function, which is simply related to relevant physical quantities. As an illustration, we specifically consider a model that cannot be solved directly by the conventional IPDF method and that can be viewed as a generalization of the *voter* model and/or as an *epidemic* model. We also consider the *reversible* diffusion-coagulation model with input of particles and determine other reaction-diffusion models that can be mapped onto the latter via suitable *similarity transformations*. Finally we study the problem of the propagation of a wave front from an inhomogeneous initial configuration and note that the mean-field scenario predicted by Fisher's equation is not valid for the one-dimensional (microscopic) models under consideration.

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I. INTRODUCTION

Reaction-diffusion models (RD) play an important role in the description of classical interacting many-particle non-equilibrium systems and have been extensively investigated in the last decade [1,2]. Often these systems have been treated by mean-field techniques (e.g., rate equations) that give rise to nonlinear partial differential equations (such as, e.g., the Fisher equation [3]). The latter represent difficult mathematical problems: e.g., the Fisher equation cannot, in general, be solved exactly. The mean-field methods can accurately describe the behavior of RD systems in *higher dimensions*, where the correlations do not dramatically change the physics of the models. However, in one spatial dimension where the fluctuations play a crucial role, these mean-field treatments fail. In this sense, a satisfactory understanding of RD models in lower dimensions requires *exact solutions*, which are scarce. In some cases, however, some RD models are known to be solvable. These cases can essentially be classified into four categories: (i) models for which the equations of motion of correlation functions are closed [4]; (ii) the *free-fermion* models [5] (or systems that can be mapped onto the latter, see [2,6]); (iii) some other one-dimensional RD models can be solved by the *matrix ansatz* method [7] first introduced to study the steady states of the asymmetric exclusion process and which has been extended to other multispecies RD models where the total number of particles is conserved, a dynamical version of the matrix ansatz [8] has also been proposed to study the dynamical properties of the models for which the equations are closed (on periodic as well as open chains); (iv) some other one-dimensional models can be solved by the empty-interval method, also called the interparticle distribution function (IPDF) method [9–12], first introduced for the study of the diffusion-coagulation model. The solution of various one-dimensional RD models have been obtained from the diffusion-coagulation model via *similarity transformations* [2,13]. It has been established that the latter solvable models correspond to free-fermion systems [2].

The purpose of this work is to present a generalization of the IPDF method and to apply this technique to solve a class of one-dimensional stochastic models that cannot be mapped onto free-fermion systems. In fact much attention has been given to free-fermion systems in various contexts (using *fermionic* algebra [5] or via the *traditional* IPDF method, in the continuum limit [9,10] as well as on discrete lattice [11,13]). The situation is different for the models considered here, for which only a few exact results are known.

The paper is organized as follows. In the next section we briefly review the formalism that we employ. In Sec. III we introduce the string function, which is the key to our analysis and determine the constraints necessary to have solvable situations. In Sec. IV we solve the general equations of motion for the string functions of reaction-diffusion models that cannot be mapped onto free-fermion systems. The latter provides the exact expression of the density and the instantaneous nearest-neighbor (two-point) correlation function. We also present an approximative, and, recursive scheme to compute the (other) instantaneous two-point correlation functions. In Sec. V, we specifically consider a model with branching and coagulation reactions that cannot be solved directly by the traditional IPDF method. In Sec. VI, we solve a reversible diffusion-coagulation model with external input of particles. In Sec. VII, we take advantage of the results of the previous section to solve other related models via similarity transformations. In Sec. VIII we study, for the models introduced and solved in Secs. V and VI, the problem of the propagation of a wave front starting from an inhomogeneous initial state and observe that the mean-field scenario predicted by *Fisher's equation* [3,14,15] is not valid at the microscopic level. Finally, Sec. IX is devoted to the conclusion.

II. THE FORMALISM

Before generalizing the IPDF method, it is useful to briefly review the so-called *stochastic Hamiltonian* formalism .

It is known that models of stochastic hard-core particles are soluble on some manifold on which the equations of motion of their correlation functions close [4]. In this work, we concentrate on one-dimensional bimolecular single-species reaction-diffusion systems.

Consider a periodic chain with L sites (labeled from 1 to L). On the lattice, local bimolecular reactions between single-species particles $\mathcal{A}$, with a hard core, take place. Each site can be empty (denoted by the symbol 0) or occupied at most by a particle of type $\mathcal{A}$ denoted in the following by the index 1 . The reactions occurring on the sites j and $j+1$ are specified by the transition rates, which here are assumed to be *site dependent*, according to $\Gamma_{\alpha\beta}^{\gamma\delta}$, where

$$\alpha,\beta,\gamma,\delta=0,1:\ \forall\ (\alpha,\beta)\neq(\gamma,\delta),\quad \Gamma_{\alpha\beta}^{\gamma\delta}:\ \alpha+\beta\rightarrow\gamma+\delta.$$

Probability conservation implies

$$\Gamma_{\alpha\beta}^{\alpha\beta}=-\sum_{(\alpha,\beta)\neq(\alpha',\beta')}\Gamma_{\alpha\beta}^{\alpha'\beta'}\ \text{and}\ \Gamma_{\alpha\beta}^{\gamma\delta}\geqslant 0,\ \forall\ (\alpha,\beta)\neq(\gamma,\delta).$$

For example, the rate Γ_{11}^{10} corresponds to the reaction $\mathcal{A}\mathcal{A}\rightarrow\mathcal{A}\varnothing$ and $\Gamma_{11}^{11}=-(\Gamma_{11}^{10}+\Gamma_{11}^{01}+\Gamma_{11}^{00})$.

The state of the system is represented by the ket $|P(t)\rangle=\Sigma_{\{n\}}P(\{n\},t)|\{n\}\rangle$, where the sum runs over the 2^L configurations. At site i the local state is specified by the ket $|n_i\rangle=(10)^T$ if the site i is empty and $|n_i\rangle=(01)^T$ if the site i is occupied by a particle of type $\mathcal{A}(1)$.

It is by now well established that a master equation can be rewritten formally as an imaginary time Schrödinger equation: $(\partial/\partial t)|P(t)\rangle=-H|P(t)\rangle$, where H is the stochastic Hamiltonian that governs the dynamics of the system. In general, it is neither Hermitian nor normal. Its construction from the master equation is a standard procedure (see, e.g., [1,2]) The evolution operator $H=\Sigma_{j=1}^{L}H_{j,j+1}$ acts locally on two adjacent sites, with

$$-H_{j,j+1}=\begin{pmatrix}\Gamma_{00}^{00} & \Gamma_{01}^{00} & \Gamma_{10}^{00} & \Gamma_{11}^{00}\\ \Gamma_{00}^{01} & \Gamma_{01}^{01} & \Gamma_{10}^{01} & \Gamma_{11}^{01}\\ \Gamma_{00}^{10} & \Gamma_{01}^{10} & \Gamma_{10}^{10} & \Gamma_{11}^{10}\\ \Gamma_{00}^{11} & \Gamma_{01}^{11} & \Gamma_{10}^{11} & \Gamma_{11}^{11}\end{pmatrix},$$

where the same notations as in Refs. [4,16,17] have been used. Probability conservation implies that each column in the above representation sums up to zero.

The *left vacuum* $\langle\tilde{\chi}|$, which is defined as $\langle\tilde{\chi}|\equiv\Sigma_{\{n\}}\langle\{n\}|$, locally has the representation $\langle\tilde{\chi}|=(11)\otimes(11)$ with the property $\langle\tilde{\chi}|H_{j,j+1}=0$.

Below we shall assume an initial state $|P(0)\rangle$ and investigate the expectation value of an operator O (observables such as density, etc.): $\langle O\rangle(t)\equiv\langle\tilde{\chi}|Oe^{-Ht}|P(0)\rangle$. For general s-species bimolecular reaction-diffusion systems, there are $(s+1)^4$ possible rates that have to fulfill the $(s+1)^2$ probability conservation constraints. Thus general s-species bimolecular reaction-diffusion systems are characterized by $(s+1)^4-(s+1)^2$ independent parameters [16,17]. If one imposes on these parameters $2s^3$ appropriate constraints, the equation of motion of correlation functions close and the system is formally soluble in arbitrary dimensions. Here, we focus on the case $s=1$, and so we have $16-4=12$ independent rates and two closure constraints.

For single-species bimolecular processes, with the notations

$$\begin{aligned}A_1&\equiv\Gamma_{00}^{01}+\Gamma_{00}^{11},\quad B_1\equiv\Gamma_{10}^{01}+\Gamma_{10}^{11}-\Gamma_{00}^{01}-\Gamma_{00}^{11},\\ C_1&\equiv\Gamma_{01}^{00}+\Gamma_{01}^{10}+\Gamma_{00}^{01}+\Gamma_{00}^{11},\\ D_1&\equiv C_1-(\Gamma_{10}^{01}+\Gamma_{10}^{11}+\Gamma_{11}^{00}+\Gamma_{11}^{10}),\\ A_2&\equiv\Gamma_{00}^{10}+\Gamma_{00}^{11},\quad B_2\equiv\Gamma_{01}^{10}+\Gamma_{01}^{11}-\Gamma_{00}^{10}-\Gamma_{00}^{11},\\ C_2&\equiv\Gamma_{10}^{00}+\Gamma_{10}^{01}+\Gamma_{00}^{10}+\Gamma_{00}^{11},\\ D_2&\equiv C_2-(\Gamma_{01}^{10}+\Gamma_{01}^{11}+\Gamma_{11}^{00}+\Gamma_{11}^{01}),\end{aligned}\tag{1}$$

the closure constraints are the following [4]:

$$\begin{aligned}&D_2=0\Rightarrow\Gamma_{00}^{10}+\Gamma_{00}^{11}-(\Gamma_{11}^{00}+\Gamma_{11}^{01})=\Gamma_{01}^{10}+\Gamma_{01}^{11}-(\Gamma_{10}^{00}+\Gamma_{10}^{01}),\\ &D_1=0\Rightarrow\Gamma_{00}^{01}+\Gamma_{00}^{11}-(\Gamma_{11}^{00}+\Gamma_{11}^{10})=\Gamma_{10}^{01}+\Gamma_{10}^{11}-(\Gamma_{01}^{00}+\Gamma_{01}^{10}),\end{aligned}\tag{2}$$

with the help of the relationships [4],

$$\begin{aligned}-\langle n_mH_{m-1,m}\rangle&=A_1+B_1\langle n_{m-1}\rangle-C_1\langle n_m\rangle+D_1\langle n_{m-1}n_m\rangle,\\ -\langle n_mH_{m,m+1}\rangle&=A_2+B_2\langle n_{m+1}\rangle-C_2\langle n_m\rangle+D_2\langle n_mn_{m+1}\rangle.\end{aligned}\tag{3}$$

The equation of motion of the density at site m is (on a periodic chain)

$$\begin{aligned}\frac{d}{dt}\langle n_m\rangle(t)=\frac{d}{dt}\langle\tilde{\chi}|n_me^{-Ht}|P(0)\rangle&=A_1+A_2+B_1\langle n_{m-1}\rangle(t)\\ &+B_2\langle n_{m+1}\rangle(t)-(C_1+C_2)\langle n_m\rangle(t)\\ &+D_1\langle n_{m-1}n_m\rangle(t)+D_2\langle n_mn_{m+1}\rangle(t).\end{aligned}\tag{4}$$

In order to illustrate the physical meaning of the models studied in this work, let us consider the latter at the mean-field level (in the continuum limit), i.e., we assume first $\langle n_x(t)\rangle\rightarrow\rho_{MF}(x,t)$ and $\langle n_xn_{x\pm1}\rangle(t)\simeq[\rho_{MF}(x,t)]^2$. At this level of approximation, we note (see Sec. VIII) that the equation of motion (4) of some *microscopic* reaction-diffusion models studied in this work, provided that $D_1<0$ and $D_2<0$, is a nonlinear partial-differential equation of Fisher type [3,15,14],

$$\frac{\partial}{\partial t}\tilde{\rho}_{MF}(x,t)=B\frac{\partial^2}{\partial x^2}\tilde{\rho}_{MF}(x,t)+k_1\tilde{\rho}_{MF}(x,t)-k_2[\tilde{\rho}_{MF}(x,t)]^2,\tag{5}$$

with $\tilde{\rho}_{MF}(x,t)\equiv\rho_{MF}(x,t)-\phi$. When $A_1=A_2=A$, $B_1=B_2=B$, $C_1=C_2=C$, and $D_1=D_2=D<0$, we have $2D\phi\equiv(C-B)+\sqrt{(B-C)^2-4AD}$, $k_1\equiv2\sqrt{(B-C)^2-4AD}>0$, which denotes the constant describing the growth and $k_2\equiv-2D>0$ is the constant describing the saturation according to the local dynamics [3,15,14].

The Fisher equation (5) admits two homogeneous steady states, namely, $\tilde{\rho}_{MF}(x,\infty)\equiv\tilde{\rho}_{MF}(\infty)=k_1/k_2$, which is stable, and another unstable steady state: $\tilde{\rho}_{MF}(\infty)=0$.

Although the Fisher equation (5) cannot be solved exactly, it is known [15,14] that the approach towards the steady state from inhomogeneous initial states [e.g., $\tilde{\rho}_{MF}(x,0)=(k_1/k_2)\Theta(x_0-x)$, where $\Theta(x')=1$ if $x'>0$ and $\Theta(x')=0$ otherwise] is characterized by a *wave front*, $\tilde{\rho}_{MF}(x,t)=f(x-ct)$, propagating with a celerity $c\geqslant 2\sqrt{k_1 B}$ and satisfying the nonlinear differential equation

$$B\frac{d^2}{dz^2}f(z)+c\frac{d}{dz}f(z)+k_1 f(z)-k_2[f(z)]^2=0\text{[15]}.$$

In this work we obtain the exact expression for the density from the N-body description of some reaction-diffusion models (on finite and periodic lattice), for which $D_1=D_2\neq 0$ and for which, the mean-field description in the continuum limit is given by a nonlinear partial-differential equation of the Fisher type (5). Therefore, with help of the (microcopic) exact results obtained in Secs. V and VI, we are able, in Sec. VIII, to discuss the validity of the Fisher's mean-field description.

III. THE STRING FUNCTION

In this section we introduce the quantity, which is the key to our analysis, i.e., the string function $S_{x,y}(t)$. We also derive the constraints for which the equation of motion of the latter is a closed hierarchy. In the sequel, we solve the latter providing the density of particles.

Instead of considering the standard empty-interval function [9–13], we focus here on the more general form ($1\leqslant x\leqslant y\leqslant x+L$),

$$S_{x,y}(t)\equiv\langle(a-bn_x)(a-bn_{x+1})\cdots(a-bn_{y-2})\times(a-bn_{y-1})\rangle(t), \tag{6}$$

where a and b are nonvanishing numbers.

This expression reduces to the empty-interval function when $a=b=1$ [9–13]. Hereafter, we will derive the equation of motion of the quantity $S_{x,y}(t)$ and determine which constraints are necessary and sufficient to close the latter. An alternative approach would consist in considering the empty-interval function [with $a=b=1$ in Eq. (6)] and obtaining a solution of some related reaction-diffusion model via a similarity transformation. This approach has been extensively studied for the free-fermion models [11,13] where solutions of RD systems are obtained from the solution of the (free-fermion) diffusion-coagulation model. The latter approach is investigated in Sec. VII. For $1\leqslant x\leqslant y<x+L$, the equation of motion of $S_{x,y}(t)$ reads

$$\begin{aligned}\frac{d}{dt}S_{x,y}(t)=&-\langle(a-bn_x)H_{x-1,x}(a-bn_{x+1})\cdots\\&\times(a-bn_{y-1})\rangle(t)-\langle(a-bn_x)\\&\times(a-bn_{x+1})\cdots(a-bn_{y-1})H_{y-1,y}\rangle(t)\\&-\sum_{j=x}^{y-2}\langle(a-bn_x)(a-bn_{x+1})\cdots\\&\times(a-bn_{y-1})H_{j,j+1}\rangle(t).\end{aligned} \tag{7}$$

If the following five constraints are fulfilled, the dynamics of $S_{x,y}(t)$ is described by a closed hierarchy of equations as follows:

$$(1)\quad aD_1=-bB_1;$$

$$(2)\quad aD_2=-bB_2;$$

$$\begin{aligned}(3-5)\quad&\Gamma_{00}^{00}+\left(\frac{a-b}{a}\right)(\Gamma_{00}^{10}+\Gamma_{00}^{01})+\left(\frac{a-b}{a}\right)^2\Gamma_{00}^{11}\\&=\frac{a}{b}(\Gamma_{00}^{00}-\Gamma_{10}^{00})-\left(\frac{a-b}{b}\right)(\Gamma_{10}^{01}+\Gamma_{10}^{10}-\Gamma_{00}^{10}-\Gamma_{00}^{01})\\&\quad-\frac{(a-b)^2}{ab}(\Gamma_{10}^{11}-\Gamma_{00}^{11})\\&=\frac{a}{b}(\Gamma_{00}^{00}-\Gamma_{01}^{00})-\left(\frac{a-b}{b}\right)(\Gamma_{01}^{01}+\Gamma_{01}^{10}-\Gamma_{00}^{10}-\Gamma_{00}^{01})\\&\quad-\frac{(a-b)^2}{ab}(\Gamma_{01}^{11}-\Gamma_{00}^{11})\\&=\left(\frac{a}{b}\right)^2(\Gamma_{11}^{00}+\Gamma_{00}^{00}-\Gamma_{10}^{00}-\Gamma_{01}^{00})+\frac{a(a-b)}{b^2}\\&\quad\times(\Gamma_{11}^{10}+\Gamma_{11}^{01}+\Gamma_{00}^{10}+\Gamma_{00}^{01}-\Gamma_{01}^{10}-\Gamma_{10}^{01}-\Gamma_{10}^{10}-\Gamma_{01}^{01})\\&\quad+\left(\frac{a-b}{b}\right)^2(\Gamma_{11}^{11}+\Gamma_{11}^{00}-\Gamma_{10}^{11}-\Gamma_{01}^{11}).\end{aligned} \tag{8}$$

The interesting point is that these five constraints are generally independent of the previous ones, Eq. (2). Therefore, in general, models that are solvable via the approach presented here are not on the ten-parameter manifold described by Eq. (2) where the equation of motion of correlation functions is closed.

When the constraints (8) are fulfilled, the equation of motion of the string function $S_{x,y}(t)$ on a periodic lattice of L sites is the following (for $aC_1 \neq bA_1$ and $aC_2 \neq bA_2$).[1]

$$\frac{d}{dt}S_{x,y}(t) = (aC_1 - bA_1)S_{x+1,y}(t) + (aC_2 - bA_2)S_{x,y-1}(t) - \frac{D_1}{b}S_{x-1,y}(t) - \frac{D_2}{b}S_{x,y+1}(t) - (B_1 + B_2 + C_1 + C_2)S_{x,y}(t) + \left[\Gamma_{00}^{00} + \left(\frac{a-b}{a}\right)(\Gamma_{00}^{10} + \Gamma_{00}^{01}) + \left(\frac{a-b}{a}\right)^2 \Gamma_{00}^{11}\right](y-x-1)S_{x,y}(t)$$

$$(1 \leqslant x < y < x+L), \quad (9)$$

$$\frac{d}{dt}S_{x,x+L}(t) = L\left[\Gamma_{00}^{00} + \left(\frac{a-b}{a}\right)(\Gamma_{00}^{10} + \Gamma_{00}^{01}) + \left(\frac{a-b}{a}\right)^2 \Gamma_{00}^{11}\right]S_{x,x+L}(t),$$

$$S_{x,x}(t) = 1,$$

where the boundary condition $S_{x,x}(t)=1$ is obtained from the requirement that $(d/dt)S_{x,x+1}(t) = -b(d/dt)\langle n_x(t)\rangle$.

Let us note that when $a=b=1$, we recover the usual constraints of the IPDF method: $\Gamma_{10}^{01}=\Gamma_{11}^{01}$, $\Gamma_{01}^{10}=\Gamma_{11}^{10}$, $\Gamma_{10}^{00} = \Gamma_{01}^{00} = \Gamma_{11}^{00} = 0$.

IV. SOLUTION OF THE EQUATION OF MOTION OF THE STRING FUNCTION

The equations of motion for the string function have been intensively studied for free-fermion systems both in the continuum limit [9,10] and on discrete lattices (periodic and open boundary conditions) [11–13]. However, for systems that cannot be mapped onto free-fermion systems, a few results have been obtained: Doering and ben-Avraham [9] have obtained the stationary concentration and the relaxation spectrum of a reversible diffusion-coagulation model on an infinite lattice in the continuum limit. Later, Peschel *et al.* [12] have studied the relaxation spectrum of the Fourier transform of the string function on a periodic lattice with help of the *conventional* IPDF method.

It is useful to introduce the following notations:

$$\frac{\alpha_1}{2} \equiv aC_1 - bA_1 \neq 0, \quad \frac{\alpha_2}{2} \equiv aC_2 - bA_2 \neq 0,$$

$$\frac{\beta_1}{2} \equiv -\frac{D_1}{b} = \frac{B_1}{b}, \quad \frac{\beta_2}{2} \equiv -\frac{D_2}{b} = \frac{B_2}{b},$$

$$\delta \equiv -\left[\Gamma_{00}^{00} + \left(\frac{a-b}{a}\right)(\Gamma_{00}^{10} + \Gamma_{00}^{01}) + \left(\frac{a-b}{a}\right)^2 \Gamma_{00}^{11}\right],$$

$$\gamma + \delta \equiv B_1 + B_2 + C_1 + C_2. \quad (10)$$

Hereafter, we solve the equation of motion (9) for the case $\delta \neq 0$ ($\delta = 0$ corresponds to the free-fermion case), $D_1 \neq 0$, and $D_2 \neq 0$ with the additional condition, $\alpha_2 = \alpha_1 \equiv \alpha \neq 0$ and $\beta_1 = \beta_2 \equiv \beta$, which corresponds to the restriction for *unbiased* systems.

We also introduce the following auxiliary function: $S_{x,y}(t) = \mu^{y-x}\mathcal{R}_{x,y}(t)$, where we choose $\mu \equiv \sqrt{\alpha_1/\beta_2} = \sqrt{\alpha/\beta}$ and $q = \sqrt{\alpha_1\beta_2} = \sqrt{\alpha\beta}$, and we solve Eq. (9) for the general case where $q \neq 0$ (notice[2] that $D_1 = D_2 = 0 \Rightarrow q = 0$). With these notations, the equation of motion (9) becomes

$$\frac{d}{dt}\mathcal{R}_{x,y}(t) = \frac{q}{2}\sum_{e=\pm 1}\{\mathcal{R}_{x+e,y}(t) + \mathcal{R}_{x,y+e}(t)\} - \gamma\mathcal{R}_{x,y}(t) - (y-x)\,\delta\mathcal{R}_{x,y}(t) \quad (1 \leqslant x < y < x+L), \quad (11)$$

$$\frac{d}{dt}\mathcal{R}_{x,x+L}(t) = -L\,\delta\mathcal{R}_{x,x+L}(t),$$

$$\mathcal{R}_{x,x}(t) = 1.$$

The stationary solution of the system (11) is obtained with help of the properties of Bessel functions of first and second kind, respectively, $J_\nu(z)$ and $Y_\nu(z)$ [19]. In fact, the structure of Eq. (11) for $1 \leqslant x < y < x+L$ suggests the ansatz

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[1] If $aC_1 = bA_1$ and $aC_2 = bA_2$, $b/a = C_1/A_1 = C_2/A_2$ and the equation of motion of $S_{x,y}$ reads

$$\frac{d}{dt}S_{x,y}(t) = -\frac{D_1}{b}S_{x-1,y}(t) - \frac{D_2}{b}S_{x,y+1}(t) - (\gamma+\delta)S_{x,y}(t) - (y-x-1)\,\delta S_{x,y} \quad (1 \leqslant x < y < x+L)$$

$$\frac{d}{dt}S_{x,x+L}(t) = -L\,\delta S_{x,x+L}(t),$$

where

$$\gamma + \delta \equiv B_1 + B_2 + C_1 + C_2 \text{ and}$$

$$\delta \equiv -\left[\Gamma_{00}^{00} + \left(\frac{a-b}{a}\right)(\Gamma_{00}^{10} + \Gamma_{00}^{01}) + \left(\frac{a-b}{a}\right)^2 \Gamma_{00}^{11}\right].$$

An example of a model in which dynamics is described by such a system of equations (with $a=b$) is the random sequential adsorbtion process of dimers, $\varnothing\varnothing \to \mathcal{A}\mathcal{A}$ [18].

[2] The case where $D_1 = D_2 = 0$ corresponds to the situation where the equations of motion of *all* the correlation functions close [see Eqs. (2) and (4)]. For a translationally invariant system with initial density $\rho(0)$and with $B_1 + B_2 \neq C_1 + C_2$, the density simply reads

$$\langle n_j(t)\rangle = \frac{A_1+A_2}{C_1+C_2-(B_1+B_2)} + \left[\rho(0) - \frac{A_1+A_2}{C_1+C_2-(B_1+B_2)}\right] \times \exp[-\{C_1+C_2-(B_1+B_2)\}t].$$

$$\mathcal{R}_{x,y}(\infty)=\tilde{A}_L J_{y-x+\omega}(2q/\delta)+\tilde{B}_L Y_{y-x+\omega}(2q/\delta).$$

Inserting this expression into Eq. (11), we obtain $\omega=\gamma/\delta$. Therefore, we have

$$\mathcal{R}_{x,y}(\infty)=\tilde{A}_L J_{y-x+\gamma/\delta}(2q/\delta)+\tilde{B}_L Y_{y-x+\gamma/\delta}(2q/\delta). \tag{12}$$

The quantities $\tilde{A}_L$ and $\tilde{B}_L$ are determined with the help of the boundary conditions:

$$\mathcal{R}_{x,x}(\infty)=1=\tilde{A}_L J_{\gamma/\delta}(2q/\delta)+\tilde{B}_L Y_{\gamma/\delta}(2q/\delta)$$

and

$$\mathcal{R}_{x,x+L}(\infty)=\tilde{A}_L J_{L+\gamma/\delta}(2q/\delta)+\tilde{B}_L Y_{L+\gamma/\delta}(2q/\delta)=0.$$

It follows that

$$\tilde{A}_L=-\frac{Y_{L+\gamma/\delta}\left(\frac{2q}{\delta}\right)}{J_{L+\gamma/\delta}\left(\frac{2q}{\delta}\right)Y_{\gamma/\delta}\left(\frac{2q}{\delta}\right)-Y_{L+\gamma/\delta}\left(\frac{2q}{\delta}\right)J_{\gamma/\delta}\left(\frac{2q}{\delta}\right)},\quad \tilde{B}_L=\frac{J_{L+\gamma/\delta}\left(\frac{2q}{\delta}\right)}{J_{L+\gamma/\delta}\left(\frac{2q}{\delta}\right)Y_{\gamma/\delta}\left(\frac{2q}{\delta}\right)-Y_{L+\gamma/\delta}\left(\frac{2q}{\delta}\right)J_{\gamma/\delta}\left(\frac{2q}{\delta}\right)}, \tag{13}$$

which, with Eq. (12), provides the stationary expression for $\mathcal{R}_{x,y}(\infty)$ and thus we obtain the stationary expression for the string function,

$$S_{x,y}(\infty)=\mu^{y-x}\left[\frac{J_{L+\gamma/\delta}(2q/\delta)Y_{y-x+\gamma/\delta}(2q/\delta)-Y_{L+\gamma/\delta}(2q/\delta)J_{y-x+\gamma/\delta}(2q/\delta)}{J_{L+\gamma/\delta}(2q/\delta)Y_{\gamma/\delta}(2q/\delta)-Y_{L+\gamma/\delta}(2q/\delta)J_{\gamma/\delta}(2q/\delta)}\right]. \tag{14}$$

To solve the dynamical part of Eq. (11), we seek a solution of the form $\mathcal{R}_{x,y}(t)-\mathcal{R}_{x,y}(\infty)=\Sigma_\lambda r^\lambda_{y,x}e^{-\lambda q t}$, which leads to the following difference equation:

$$r^\lambda_{y,x+1}+r^\lambda_{y-1,x}+r^\lambda_{y,x-1}+r^\lambda_{y+1,x}+2\left[\lambda-\left\{\frac{\gamma+\delta(y-x)}{q}\right\}\right]r^\lambda_{y,x}=0 \tag{15}$$

with the boundary conditions

$$(L\delta-\lambda q)r^\lambda_{x,x+L}=0 \quad \text{and} \quad r^\lambda_{x,x}=0. \tag{16}$$

Introducing the notation $E\equiv(q\lambda-\gamma)/\delta$, Eq. (15) admits as solution

$$r^\lambda_{x,y}=\tilde{\mathcal{A}}J_{y-x-E}(2q/\delta)+\tilde{\mathcal{B}}Y_{y-x-E}(2q/\delta),$$

where $\tilde{\mathcal{A}},\tilde{\mathcal{B}}$, and the (relaxation) spectrum $\{E\}$ are determined from the boundary conditions (16), which imply

$$\begin{aligned}\tilde{\mathcal{A}}J_{-E}(2q/\delta)+\tilde{\mathcal{B}}Y_{-E}(2q/\delta)=0,\\ \tilde{\mathcal{A}}J_{L-E}(2q/\delta)+\tilde{\mathcal{B}}Y_{L-E}(2q/\delta)=0.\end{aligned} \tag{17}$$

The only nontrivial solution of this system (for which $\tilde{\mathcal{A}}\neq 0$ and $\tilde{\mathcal{B}}\neq 0$) requires

$$J_{L-E}(2q/\delta)Y_{-E}(2q/\delta)-J_{-E}(2q/\delta)Y_{L-E}(2q/\delta)=0, \tag{18}$$

or equivalently in terms of the *Lommel function* [19]:

$$R_{L-1,1-E}(2q/\delta)=0, \tag{19}$$

which admits $L-1$ zeroes [12,20,19] with degeneracy L. The latter are symmetrically distributed around $L/2$ (which is an eigenvalue if L is even). To obtain the complete set of $L(L-1)+1$ eigenvalues, i.e., the complete relaxation spectrum $\{E_i\}, i=1,\ldots,L$, of the string function (*which has not to be confused with the spectrum of the stochastic Hamiltonian H)*, one has to take into account the eigenvalue $q\lambda=L\delta$ directly obtained from the boundary condition (16). Notice that in $\{E_i\}$ the index $i=1,\ldots,L$ labels the L distinct eigenvalues forming the relaxation spectrum. In order to have some more insight into the relaxation spectrum $\{E_i\}$, $i=1,\ldots,L$, of the string-function $S_{x,y}(t)$, we can take advantage from the fact that the following eigenvalue problem:

$$\begin{aligned}(E-n)F_n=V(F_{n-1}+F_{n+1}) \quad (1\leqslant n<L),\\ F_0=F_L=0\end{aligned} \tag{20}$$

admits as eigenvalues the $(L-1)$ zeroes of the *Lommel function* $R_{L-1,1-E}(2V)=0$ [20,12]. Therefore, choosing $V=q/\delta$, the problem of determining the relaxation spectrum is reformulated as that of solving the eigenvalue problem (20), $\mathcal{M}|\mathcal{F}\rangle\rangle=E|\mathcal{F}\rangle\rangle$, where $\mathcal{M}$ is a $(L-1)\times(L-1)$ symmetric (but not Hermitian when q has an imaginary part) tridiagonal matrix and $|\mathcal{F}\rangle\rangle$ is a $(L-1)$-component column vector: $|\mathcal{F}\rangle\rangle\equiv(F_{n=1}F_2\cdots F_{L-1})^T$. The general form of the matrix $\mathcal{M}$ is the following:

$$\mathcal{M}=\begin{pmatrix} 1 & q/\delta & 0 & \dots & \dots & \dots & 0 \\ q/\delta & 2 & q/\delta & 0 & \dots & \dots & 0 \\ 0 & q/\delta & 3 & q/\delta & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \ddots & q/\delta & (L-2) & q/\delta \\ 0 & \dots & \dots & \dots & 0 & q/\delta & (L-1) \end{pmatrix}. \tag{21}$$

For small systems the $(L-1)$ distinct eigenvalues $\{E_i\}$ of $\mathcal{M}$ can be computed analytically. For $L=6$, with $V=q/\delta$, we have

$$\{E_i\}=\left\{3{,}3\pm\sqrt{\frac{5+4V^2\pm\sqrt{9+24V^2+4V^4}}{2}}\right\},$$

where we still have to take into account the additional eigenvalue $q\lambda=L\delta$. The spectrum depends on the size L of the system and this is, in particular, the case for $E^*\equiv \min_E\{E\}\equiv\epsilon_L$, the smallest eigenvalue that governs the long-time behavior of the system. For larger matrices we had to proceed numerically and for $L\gg 1$, $\epsilon_L\to\epsilon_\infty$ and E^* is a constant: $E^*=\epsilon_\infty$. For $L=6$, we have the exact result

$$\epsilon_{L=6}=3-\sqrt{\frac{5+4V^2+\sqrt{9+24V^2+4V^4}}{2}}.$$

This expression can be considered as an excellent approximation for systems of size $L\gg 1$ [see footnote (4)] and, in particular, for ϵ_∞. Numerical results show that the eigenvalues of Eq. (21) are ''close'' to the integers and thus never coincide, $\forall L$. Therefore, the long-time dynamics (of large systems, with $L\gg 1$) is governed by the eigenvalue ($V\equiv q/\delta$)

$$E^*=\epsilon_L\simeq 3-\sqrt{\frac{5+4\left(\frac{q}{\delta}\right)^2+\sqrt{9+24\left(\frac{q}{\delta}\right)^2+4\left(\frac{q}{\delta}\right)^4}}{2}}=\epsilon_{L=6}, \tag{22}$$

i.e.,

$$q\lambda^*=E^*\delta+\gamma=\epsilon_L\delta+\gamma. \tag{23}$$

This expression provides the inverse of the relaxation time of the model under consideration.

Having obtained the relaxation spectrum and the expression of $r^{\lambda}_{y,x}$, the complete expression for the string function follows as

$$\begin{aligned} S_{x,y}(t)-S_{x,y}(\infty)=\mu^{y-x}\sum_{E_i}\ & \mathcal{A}_{E_i}e^{-(E_i\delta+\gamma)t} \\ & \times[J_{y-x-E_i}(2q/\delta)Y_{L-E_i}(2q/\delta) \\ & -Y_{y-x-E_i}(2q/\delta)J_{L-E_i}(2q/\delta)]. \end{aligned} \tag{24}$$

Here for simplicity we consider the translationally invariant (but *not* necessarily uncorrelated) situation, when $S_{x,y}(t)=S_{y-x}(t)$. In this case, the coefficients $\mathcal{A}_{E_i}$ are obtained from the initial condition according to

$$\begin{aligned} \mathcal{A}_{E_i}=\sum_{j,n=1}^{L}\ & [\mathcal{N}^{-1}]_{i,j}[J_{n-E_j}(2q/\delta)Y_{L-E_j}(2q/\delta) \\ & -Y_{n-E_j}(2q/\delta)J_{L-E_j}(2q/\delta)]^*[S_n(0)-S_n(\infty)]\mu^{-n}, \end{aligned} \tag{25}$$

where $\mathcal{N}$ is a Hermitian $L\times L$ matrix [see Eq. (30)].

To clarify this point let us introduce the following vectors of the Hilbert space $\mathbf{C}^L$ (with the usual scalar product):

$$|S\rangle\rangle\equiv[\{S_1(0)-S_1(\infty)\}\mu^{-1}\cdots\{S_L(0)-S_L(\infty)\}\mu^{-L}]^T \tag{26}$$

and

$$|V_{E_j}\rangle\rangle \equiv \begin{pmatrix} J_{1-E_j}(2q/\delta)Y_{L-E_j}(2q/\delta) - Y_{1-E_j}(2q/\delta)J_{L-E_j}(2q/\delta) \\ \vdots \\ J_{L-1-E_j}(2q/\delta)Y_{L-E_j}(2q/\delta) - Y_{L-1-E_j}(2q/\delta)J_{L-E_j}(2q/\delta) \\ 0 \end{pmatrix}. \tag{27}$$

In a vectorial formulation, the coefficients $\mathcal{A}_{E_j}$ are obtained from the initial state of the system according to

$$|S\rangle\rangle = \sum_{j=1}^{L} \mathcal{A}_{E_j}|V_{E_j}\rangle\rangle. \tag{28}$$

Solving this equation, we formally obtain the expression for the coefficients $\mathcal{A}_{E_j}$,

$$\mathcal{A}_{E_j} = \sum_{j=1}^{L} [\mathcal{N}^{-1}]_{i,j}\langle\langle V_{E_j}|S\rangle\rangle, \tag{29}$$

where $\mathcal{N}$ is Hermitian $L\times L$ matrix whose entries read

$$\begin{aligned}[\mathcal{N}]_{i,j} \equiv \langle\langle V_{E_i}|V_{E_j}\rangle\rangle = \sum_{n=1}^{L} & [J_{n-E_i}(2q/\delta)Y_{L-E_i}(2q/\delta) \\ & - Y_{n-E_i}(2q/\delta)J_{L-E_i}(2q/\delta)]^* \\ & \times [J_{n-E_j}(2q/\delta)Y_{L-E_j}(2q/\delta) \\ & - Y_{n-E_j}(2q/\delta)J_{L-E_j}(2q/\delta)]. \end{aligned} \tag{30}$$

With the help of the expression of the string function (24), we can compute the exact density of particles at site x,

$$\langle n_x(t)\rangle = \frac{a - S_{x,x+1}(t)}{b}. \tag{31}$$

In the nontranslationally invariant situation, we would proceed in a similar manner, but we would have to work with vectors of the Hilbert space $\mathbf{C}^{L(L-1)+1}$. In this case we would have to take into account the degeneracy of the eigenvalues of $\mathcal{M}$ in order to compute the $L(L-1)+1$ components $\mathcal{A}_E$ appearing in Eq. (24). This is achieved by replacing E_i with $E_{i,d}$ in Eq. (24), where the index d labels the degeneracy of the eigenvalues E_i, $i=1,\ldots,L$.

From Eq. (31), we can also obtain the expression of the noninstantaneous two-point correlation functions $\langle n_x(t)n_{x_0}(0)\rangle$. We should take into account the initial state $|P'(0)\rangle\rangle \equiv n_{x_0}|P(0)\rangle\rangle$ instead of $|P(0)\rangle\rangle$ in Eq. (31).

With Eq. (24), we can also compute the *instantaneous* nearest-neighbor (two-point) correlation functions,

$$\langle n_x n_{x+1}\rangle(t) = \frac{a^2 + S_{x,x+2}(t) - a(S_{x,x+1}(t) + S_{x+1,x+2}(t))}{b^2}. \tag{32}$$

Although the present approach could be formally extended to obtain the exact and closed equations of motion of other stringlike functions, whose (eventual) resolution would provide the exact expressions of all instantaneous two-point correlation functions, in practice such equations turn out to be extremely difficult to solve. The only cases where the whole hierarchy of equation has been completely solved are the free-fermion models. On the discrete (and finite) lattice these solutions were obtained by Krebs *et al.* [11] and in the continuum limit (for the diffusion-coagulation $\mathcal{A}\mathcal{A}\to\mathcal{A}$ model) by ben-Avraham [10]. Unfortunately it is known that the latter approaches cannot be extended to systems that cannot be mapped onto free-fermion systems [11,10]. Here we prefer to take advantage of the quantities $\langle n_x(t)\rangle$ and $\langle n_x n_{x+1}(t)\rangle$ and $S_{x,y}(t)$, which we can compute exactly to obtain approximative instantaneous two-point correlation functions of the systems obeying the constraints (8), whose associated string function $S_{x,y}(t)$ obeys the equation of motion (9).

For technical convenience, we consider the translationally invariant situation [thus $\langle n_x(t)\rangle = \rho(t)$] and expanding the string function we have

$$\begin{aligned} \left(\frac{a}{b}\right)^2 & \left[a^{-(y-x)}S_{y-x}(t) + \frac{b(y-x)}{a}\rho(t)\right] \\ & = \sum_{j=1}^{y-1} (y-x-j)\langle n_{j_1} n_{j_1+j}\rangle(t) + \cdots + \left(\frac{b}{a}\right)^{m-2} \\ & \quad \times \sum_{y-x>j_1>j_2>\ldots>j_m} \langle n_{j_1} n_{j_2} \ldots n_{j_m}\rangle(t) + \cdots \\ & \quad + b^{y-x-2}\langle n_x n_{x+1}\cdots n_{y-1}\rangle(t). \end{aligned} \tag{33}$$

From Eq. (33), it is possible to obtain exact expressions relating a two-point correlation function with known quantities and higher-order correlation functions. As an illustration, let us first consider the case where $y-x=3$. Equation (33) implies

$$\begin{aligned} \langle n_x n_{x+2}\rangle(t) - \frac{b}{a}\langle n_x n_{x+1} n_{x+2}\rangle(t) \\ = \frac{S_{x,x+3}}{ab^2} - \frac{a^2}{b} + \frac{3a\rho(t)}{b} - 2\langle n_x n_{x+1}\rangle(t). \end{aligned} \tag{34}$$

For $y-x=4$, Eq. (33) with help of Eq. (34) leads to

$$\langle n_x n_{x+3}\rangle(t) - \left(\frac{b}{a}\right)^2 \langle n_x n_{x+1} n_{x+2} n_{x+3}\rangle(t)$$
$$= \frac{S_{x,x+4}(t) - 2aS_{x,x+3}(t)}{(ab)^2} + \left(\frac{a}{b}\right)^2 - \frac{2a\rho(t)}{b}$$
$$+ \langle n_x n_{x+1}\rangle(t). \qquad (35)$$

This procedure can naturally be continued for every two-point correlation functions. Therefore, for two-point correlation functions $\langle n_x n_{x+r}\rangle(t)$ of sites separated by a distance r, using recursively the relations previously derived for $\langle n_x n_{x+r-1}\rangle(t)$, $\langle n_x n_{x+r-2}\rangle(t)$, ... we obtain an equality relating $\langle n_x n_{x+r}\rangle(t)$ and an unknown $(r+1)$-point correlation function $[\langle n_x n_{x+1}\cdots n_{x+r-1} n_{x+r}\rangle(t)]$ to a combination of known quantities, as in Eqs. (34) and (35).

It is therefore possible to obtain approximative expressions for the correlation function within the truncation approximation (for r even),

$$\langle n_x n_{x+1}\cdots n_{x+r-1} n_{x+r}\rangle(t) \simeq \langle n_x n_{x+1}\rangle(t)\cdots \times \langle n_{x+r-1} n_{x+r}\rangle(t) = [\langle n_x n_{x+1}\rangle(t)]^{r/2}$$

and

$$\langle n_x n_{x+1}\ldots n_{x+r-1} n_{x+r}\rangle(t) \simeq [\langle n_x n_{x+1}\rangle(t)]^{(r-1)/2}\rho(t),$$

for r odd.

Within this mean-field-like approach, we obtain the following approximative expressions for the two-point correlation functions:

$$\langle n_x n_{x+2}\rangle(t) \simeq \frac{S_{x,x+3}(t)}{ab^2} - \left[2 + \frac{b\rho(t)}{a}\right]\langle n_x n_{x+1}\rangle(t) - \frac{3a\rho(t)}{b} - \left(\frac{a}{b}\right)^2,$$

$$\langle n_x n_{x+3}\rangle(t) \simeq \frac{S_{x,x+4}(t) - 2aS_{x,x+3}(t)}{(ab)^2} + \left[1 - \left(\frac{b}{a}\right)^2 \langle n_x n_{x+1}\rangle(t)\right] \times \langle n_x n_{x+1}\rangle(t) - \frac{2a\rho(t)}{b} + \left(\frac{a}{b}\right)^2,$$

$$\vdots \qquad (36)$$

To conclude this section, let us comment on this recursive procedure.

First of all, the *recursive* character of the method appears through the repeated use of Eq. (33) and of the relations obtained for the other two-point correlation functions. The advantages of this recursive mean-field-like method with respect to the traditional mean field are the following.

(i) The procedure of truncation appears at the level of the three-point correlation functions.

(ii) This approach is based on the explicit knowledge of the quantities $S_{y-x}(t)$, $\rho(t)$ and $\langle n_x n_{x+1}\rangle(t)$.

(iii) This method does not give rise to nonlinear partial differential equations and/or to nonlinear self-consistent equations, which are generally difficult to solve and which appear from traditional mean-field methods. Conversely, the approach presented here gives directly (after the truncation procedure) access to the (approximative) expressions of the two-point correlation functions.

It follows from the exact expression (24) of $S_{x,y}(t)$ that, for the models under consideration in this work, the related string function approaches its steady state exponentially fast, with an inverse relaxation time given by Eq. (23). This result is valid for an arbitrary initial state: the effect of the initial condition only appears through multiplicative factors $\mathcal{A}_E$. In the translationally invariant situation we have the coefficients (25); other initial conditions do not affect the exponential nature of the relaxation (24) with the inverse of relaxation time (23).

V. A MODEL THAT CANNOT BE SOLVED DIRECTLY FROM THE CONVENTIONAL IPDF METHOD

In this section we consider a model that cannot be solved directly by the conventional IPDF method. A brief account of the study of this model has recently been reported in [21]. Here we complete and develop this preliminary work.

The dynamics of the model under consideration takes place on a finite and periodic lattice. When a particle and a vacancy are adjacent to each other, a *branching reaction* can take place and the particle $\mathcal{A}$ can give birth to an offspring ($\mathcal{A}\varnothing \to \mathcal{A}\mathcal{A}$ and $\varnothing\mathcal{A} \to \mathcal{A}\mathcal{A}$) with rate $\Gamma_{10}^{11} = \Gamma_{01}^{11}$; another possible reaction is the *death* of the particle ($\mathcal{A}\varnothing \to \varnothing\varnothing$ and $\varnothing\mathcal{A} \to \varnothing\varnothing$) with rate $\Gamma_{10}^{00} = \Gamma_{01}^{00}$. When two particles are adjacent, they can *coagulate* ($\mathcal{A}\mathcal{A} \to \mathcal{A}\varnothing$ and $\mathcal{A}\mathcal{A} \to \varnothing\mathcal{A}$) with rate $\Gamma_{11}^{10} = \Gamma_{11}^{01}$. In addition, when two vacancies are adjacent, a particle can appear (*birth* process, $\varnothing\varnothing \to \mathcal{A}\varnothing$ and $\varnothing\varnothing \to \varnothing\mathcal{A}$) with rate $\Gamma_{00}^{10} = \Gamma_{00}^{01}$. The dynamics of this branching coagulation with birth and death processes (BCBD) model can be encoded by the following reactions:

$$\mathcal{A}\varnothing \to \mathcal{A}\mathcal{A} \text{ and } \varnothing\mathcal{A} \to \mathcal{A}\mathcal{A} \text{ with rate } \Gamma_{10}^{11} = \Gamma_{01}^{11},$$
$$\varnothing\varnothing \to \varnothing\mathcal{A} \text{ and } \varnothing\varnothing \to \mathcal{A}\varnothing \text{ with rate } \Gamma_{00}^{01} = \Gamma_{00}^{10},$$
$$\mathcal{A}\mathcal{A} \to \mathcal{A}\varnothing \text{ and } \mathcal{A}\mathcal{A} \to \varnothing\mathcal{A} \text{ with rate } \Gamma_{11}^{10} = \Gamma_{11}^{01},$$
$$\mathcal{A}\varnothing \to \varnothing\varnothing \text{ and } \mathcal{A}\varnothing \to \varnothing\varnothing \text{ with rate } \Gamma_{10}^{00} = \Gamma_{01}^{00}. \qquad (37)$$

It should be emphasized that in this model, the effective motion of the particles is realized by successive processes of branching, coagulation, birth, and death on neighboring pairs of lattice sites, without an *explicit* diffusion process.

The system described above can be viewed as an epidemic model where particles can spontaneously appear/disappear, have an offspring, and coagulate. It can also be viewed as a generalization of the voter model [2], where the presence/absence of particle is associated with an opinion (yes/no) and each site is associated with a human being.

According to the dynamics of the model, each individual changes his opinion at a rate proportional to the opinion of his neighbors.

For the model under consideration $A_1=A_2=A=\Gamma_{00}^{10}$, $B_1=B_2=B=\Gamma_{10}^{11}-\Gamma_{00}^{10}$, $C_1=C_2=C=\Gamma_{10}^{00}+\Gamma_{00}^{10}$, $D_1=D_2=D=\Gamma_{10}^{00}+\Gamma_{00}^{10}-(\Gamma_{10}^{11}+\Gamma_{11}^{10})$. If $D=0$, for the translationally invariant situation with an initial density $\rho(0)$ of particles, we have [when $B\neq C$, see footnote (2)]

$$\langle n_x(t)\rangle=\frac{A}{C-B}+\left[\rho(0)-\frac{A}{C-B}\right]e^{-2(C-B)t}.$$

In this section we solve, with some restrictions on the reaction rates, the above-mentioned model when $D\neq 0$, i.e., in the case where the equation of motion of the correlation functions of the model give rise to an open hierarchy (4). It has to be stressed that this model can be casted into a free-fermion form only when $\Gamma_{10}^{11}=\Gamma_{10}^{00}$ and $\Gamma_{11}^{10}=\Gamma_{00}^{10}$ (see [2] for a complete classification of free-fermion systems). Furthermore, this model cannot be solved (directly) by the traditional IPDF method (not applicable [9–12] in the presence of the processes $\mathcal{A}\varnothing\rightarrow\varnothing\varnothing;\varnothing\mathcal{A}\rightarrow\varnothing\varnothing$ and in the absence of processes $\mathcal{A}\varnothing\rightarrow\varnothing\mathcal{A};\varnothing\mathcal{A}\rightarrow\mathcal{A}\varnothing$; the latter should occur with the same rate as the coagulation rates [9–12]). The idea is to *choose* suitable a and b to close the equation of evolution of $S_{x,y}(t)$. This is achieved by imposing the following condition:

$$\frac{b}{a}=1+\frac{\Gamma_{11}^{10}}{\Gamma_{00}^{10}}>1, \tag{38}$$

and the reaction rates fulfill

$$\Gamma_{11}^{10}=\Gamma_{11}^{01}>0,\ 2\Gamma_{00}^{10}=2\Gamma_{00}^{01}\geqslant\Gamma_{10}^{11}=\Gamma_{01}^{11}>0, \text{ and}$$

$$\Gamma_{10}^{00}=\Gamma_{01}^{00}=\frac{\Gamma_{11}^{10}(2\Gamma_{00}^{10}-\Gamma_{10}^{11})}{\Gamma_{00}^{10}}\geqslant 0. \tag{39}$$

We will see that the case treated in this section [with the constraints (39)] can be obtained from the reversible model of diffusion coagulation with input of particles [reversible diffusion coagulation with input of particles (RDCI) model], solved in the next section, via a similarity transformation. In fact, in Sec. VII, we investigate a local similarity transformation that would map the general $S_{x,y}(t)$ function onto the empty-interval function (with $a=b=1$) and the RDCI model onto the present BCBD model. In the sequel we show that such a mapping exists and establish that the approaches followed in this section and in Sec. VII are equivalent.

For the model (37) with the restriction (39) and from the definitions (10), we have $\alpha=(2a\Gamma_{11}^{10}/\Gamma_{00}^{10})(\Gamma_{00}^{10}-\Gamma_{10}^{11})$, $\beta=-(2/a)(\Gamma_{00}^{10}-\Gamma_{10}^{11})$ and thus $\alpha\beta<0$, $\mu_1=\mu_2=\mu=-i[\mathrm{sgn}(\alpha)]|\alpha/\beta|^{1/2}$, and $q=i|\alpha\beta|^{1/2}$. We also have $\delta=2bA/a>0$ and [because of Eq. (39)] $0<|q|/\delta<1/2$.

The subcase $\Gamma_{10}^{11}=\Gamma_{00}^{10}$ implies $\alpha=\beta=B=D=0$ and we recover (for $C\neq 0$) $\langle n_x(t)\rangle=[a-S_{x,x+1}(t)]/b=A/C+[\langle n_x(0)\rangle-A/C]e^{-2Ct}$.

Hereafter, we focus on the more general situation where Eqs. (39) are fulfilled with $\Gamma_{10}^{11}\neq\Gamma_{00}^{10}$, and thus $\alpha\neq 0,\beta\neq 0$.

The stationary expression of the string function for this model is given by the expression (14). With help of the formula (31) and the ratio (38), we obtain the following expression for the stationary density of particles in the system:

$$\langle n_x(\infty)\rangle=\frac{1}{b}\left\{a-\mu\left[\frac{J_{L+\gamma/\delta}(2q/\delta)Y_{1+\gamma/\delta}(2q/\delta)-Y_{L+\gamma/\delta}(2q/\delta)J_{1+\gamma/\delta}(2q/\delta)}{J_{L+\gamma/\delta}(2q/\delta)Y_{\gamma/\delta}(2q/\delta)-Y_{L+\gamma/\delta}(2q/\delta)J_{\gamma/\delta}(2q/\delta)}\right]\right\}. \tag{40}$$

Similarly, with help of Eqs. (31) and (38), we obtain the stationary expression of the instantaneous nearest-neighbor correlation functions,

$$\langle n_x n_{x+1}\rangle(\infty)=\frac{2ab\langle n_x(\infty)\rangle-a^2}{b^2}+\left(\frac{\mu}{b}\right)^2\left[\frac{J_{L+\gamma/\delta}(2q/\delta)Y_{2+\gamma/\delta}(2q/\delta)-Y_{L+\gamma/\delta}(2q/\delta)J_{2+\gamma/\delta}(2q/\delta)}{J_{L+\gamma/\delta}(2q/\delta)Y_{\gamma/\delta}(2q/\delta)-Y_{L+\gamma/\delta}(2q/\delta)J_{\gamma/\delta}(2q/\delta)}\right]. \tag{41}$$

From Eqs. (40) and (41) one can check that the system under consideration is a *correlated system of interacting particles*. In fact, one can see that $\langle n_x n_{x+1}\rangle(\infty)\neq\langle n_x(\infty)\rangle\langle n_{x+1}(\infty)\rangle=[\langle n_x(\infty)\rangle]^2$, despite the fact that both steady states (40) and (41) are translationally invariant, the stationary distribution is correlated, which is due to the interacting character (hard core) of the particles. We should emphasize that the presence of correlations in the stationary distribution is specific to the class of models considered here that cannot be mapped onto free-fermion systems[3] (see also the model of the next section).

[3]In fact, for the free-fermion systems such as the diffusion limited with pair annihilation and creation model [5] and the related diffusion-coagulation models [9–11], it has been shown, for translationally invariant systems, that $\langle n_x n_{x+r}\rangle(\infty)=[\langle n_x(\infty)\rangle]^2$, $\forall\ r>0$.

To study the dynamical properties of the model, we need the relaxation spectrum of the string function. As established in the previous section, the latter is obtained as the set of zeros of the following Lommel function: $R_{L-1,1-E}(2i|q|/\delta)=0$, where $E\equiv(q\lambda-\gamma)/\delta$. Solving the associate eigenvalue problem (20) (in this case, the matrix $\mathcal{M}$, see Eq. (21), is not Hermitian).

For small systems the $(L-1)$ distinct eigenvalues $\{E_i\}$ of Eq. (21) can be computed analytically. For $L=6$, we have $(V\equiv q/\delta)$

$$\{E\}=\left\{3,3\pm\sqrt{\frac{5+4V^2\pm\sqrt{9+24V^2+4V^4}}{2}}\right\}.$$

For larger matrices we had to proceed numerically. Our analysis (based on the spectrum of large matrices with $L\leqslant 1000$), shows that the spectrum $\{E\}$ (and therefore $\{q\lambda\}$) is always real and symmetric around $L/2$, which is an eigenvalue when L is even. The other eigenvalues are not generally integers, but for the central part of the spectrum (for eigenvalues close of $L/2$), the eigenvalues approach integer values. This is not the case at the extremities of the spectrum. In particular, the smallest eigenvalue $E^*=\min_E\{E\}$ is not an integer and depends on the size of the system, $E^*=\epsilon_L>1$. However, for $L\gg 1$, $\epsilon_L\to\epsilon_\infty$, and E^* is a constant, $E^*=\epsilon_\infty>1$. For $L=6$, we have the exact result

$$\epsilon_{L=6}=3-\sqrt{\frac{5+4V^2+\sqrt{9+24V^2+4V^4}}{2}},$$

with $1<\epsilon_{L=6}<3-\sqrt{2+\frac{1}{4}\sqrt{13}}$. This expression can be considered as an excellent approximation for systems of size $L\gg 1$ and, in particular, for ϵ_∞.[4]

Therefore, the long-time dynamics (of large systems, with $L\gg 1$) is governed by the eigenvalue

$$E^*=\epsilon_L\simeq 3-\sqrt{\frac{5+4V^2+\sqrt{9+24V^2+4V^4}}{2}},$$

i.e.,

$$\begin{aligned}q\lambda^*=E^*\delta+\gamma=\epsilon_L\delta+\gamma=2\Bigg[&\frac{\Gamma_{00}^{10}\Gamma_{10}^{11}+\Gamma_{11}^{10}(2\Gamma_{00}^{10}-\Gamma_{10}^{11})}{\Gamma_{00}^{10}}\\&+(\epsilon_L-1)(\Gamma_{00}^{10}+\Gamma_{11}^{10})\Bigg]\\&>2\Gamma_{10}^{11}\\&\geqslant 0.\end{aligned}\tag{42}$$

[4]As an illustration, for the case $\Gamma_{00}^{10}=3/10$, $\Gamma_{10}^{11}=1/2$, $\Gamma_{11}^{10}=1$, and $\Gamma_{10}^{00}=1/3$, with the expression above, we obtain (analytically) $\epsilon_{L=6}=1.082\,333\,768\,3$. For larger systems ($L=10,25,40,1000$), we obtain numerically (with an accuracy of 10^{-10}) $\epsilon_{10}=\epsilon_{25}=\epsilon_{40}=\epsilon_{1000}=1.082\,333\,769\,7$.

Equation (42) provides the inverse of the relaxation time of the system.

The dynamical approach towards the steady state of the density is obtained from the dynamical expression (24) of the string function, according to formula (31) and with the ratio (38), we obtain

$$\begin{aligned}\langle n_x(t)\rangle-\langle n_x(\infty)\rangle=\left(\frac{\mu}{b}\right)\sum_{E_i}&\mathcal{A}_{E_i}e^{-(E_i\delta+\gamma)t}\\&\times[J_{1-E_i}(2q/\delta)Y_{L-E_i}(2q/\delta)\\&-Y_{1-E_i}(2q/\delta)J_{L-E_i}(2q/\delta)],\end{aligned}\tag{43}$$

where the coefficients $\mathcal{A}_{E_i}$ have been computed for the translationally invariant situation in Eqs. (25)–(30).

With help of Eqs. (24) and (31), we obtain the expression of the dynamical approach of the instantaneous nearest neighbor to its steady state (41),

$$\begin{aligned}&\langle n_xn_{x+1}\rangle(t)-\langle n_xn_{x+1}\rangle(\infty)\\&\quad=\frac{a}{b}\{\langle n_x(t)\rangle+\langle n_{x+1}(t)\rangle-2\langle n_x(\infty)\rangle\}+\left(\frac{\mu}{b}\right)^2\\&\qquad\times\sum_{E_i}\mathcal{A}_{E_i}e^{-(E_i\delta+\gamma)t}[J_{2-E_i}(2q/\delta)Y_{L-E_i}(2q/\delta)\\&\qquad-Y_{2-E_i}(2q/\delta)J_{L-E_i}(2q/\delta)].\end{aligned}\tag{44}$$

From Eqs. (24), (43), and (44) and with the help of Eqs. (33)–(36), we can also obtain the approximative expression of the other two-point correlation functions.

The result (43) can be extended to the case of the noninstantaneous two-point correlation functions $\langle n_x(t)n_{x_0}(0)\rangle$. It suffices to replace in Eq. (43) the coefficients $\mathcal{A}_{E_i}$ by those obtained from the initial condition $\langle\{\Pi_{j=x}^{y-1}[a-bn_j(0)]\}n_{x_0}(0)\rangle$, instead of $S_{y-x}(0)$.

Let us now mention the long-time behavior of quantities computed above. For the sake of simplicity we consider the translationally invariant situation, in the regime where $E^*t\gg 1$ [E^* is the smallest eigenvalue (42)] we have

$$\begin{aligned}\langle n_x(t)\rangle-\langle n_x(\infty)\rangle\sim\left(\frac{\mu}{b}\right)&\mathcal{A}_{E^*}e^{-(E^*\delta+\gamma)t}\\&\times[J_{1-E^*}(2q/\delta)Y_{L-E^*}(2q/\delta)\\&-Y_{1-E^*}(2q/\delta)J_{L-E^*}(2q/\delta)],\end{aligned}\tag{45}$$

$$\langle n_x n_{x+1}\rangle(t)-\langle n_x n_{x+1}\rangle(\infty)$$
$$\sim \frac{\mu^2 \mathcal{A}_{E^*}}{b^2} e^{-(E^*\delta+\gamma)t}\left[J_{2-E^*}\left(\frac{2q}{\delta}\right) Y_{L-E^*}\left(\frac{2q}{\delta}\right) - Y_{2-E^*}\left(\frac{2q}{\delta}\right) J_{L-E^*}\left(\frac{2q}{\delta}\right)\right] + \frac{2a\mu\mathcal{A}_{E^*}}{b^2} e^{-(E^*\delta+\gamma)t} \times\left[J_{1-E^*}\left(\frac{2q}{\delta}\right) Y_{L-E^*}\left(\frac{2q}{\delta}\right) - Y_{1-E^*}\left(\frac{2q}{\delta}\right) \times J_{L-E^*}\left(\frac{2q}{\delta}\right)\right]. \quad (46)$$

It follows from the exact results (42)–(46) that the density $\langle n_x(t)\rangle$ and the two-point correlation functions $\langle n_x(t) n_{x_0}(0)\rangle$ (noninstantaneous) and $\langle n_x n_{x+1}\rangle(t)$ (instantaneous) approach the steady state exponentially fast, as the string function $S_{x,y}(t)$, with an inverse of relaxation time given by Eq. (42). In addition, in the sense of the approximative scheme (33)–(36), these results are expected also to be valid for the more general correlation functions $\langle n_x n_{x+r}\rangle(t)(r>1)$ and for arbitrary initial conditions: the initial state only affects the multiplicative factors $\mathcal{A}_E$ of Eqs. (43) and (44) [for translationally invariant systems the $\mathcal{A}_E$ are given by Eq. (25)]. These statements are supported by the study of the subcase $\Gamma_{10}^{11}=\Gamma_{00}^{10}$ (with $C\neq 0$), which is solvable by conventional methods and in which the dynamics is expected to be *qualitatively* the same as the more general case considered here (especially for $|\Gamma_{10}^{11}-\Gamma_{00}^{10}|\ll 1$). For this subcase the density has been computed previously (for arbitrary initial condition) and the two-point correlation functions read

$$\langle n_x(t) n_{x_0}(0)\rangle = \frac{A}{C} + \left[\langle n_x n_{x_0}\rangle(0) - \frac{A}{C}\right] e^{-2Ct}$$

and

$$\langle n_x n_y\rangle(t) = \left(\frac{A}{C}\right)^2 (1-e^{-2Ct}) + \langle n_x n_y\rangle(0) e^{-4Ct} + \frac{A}{C}\left[\langle n_x(0)\rangle + \langle n_y(0)\rangle - \frac{A}{C}\right](e^{-2Ct} - e^{-4Ct}), \quad (y\neq x).$$

VI. SOLUTION OF A REVERSIBLE DIFFUSION-COAGULATION MODEL WITH INPUT OF PARTICLES

In this section we study a model of reversible diffusion coagulation with input of particles, which can be solved by the conventional IPDF method. Particles can jump (provided that the arrival site was previously empty) to the right and the left with rate $\Gamma_{10}^{01}=\Gamma_{01}^{10}>0$. We assume also that two adjacent particles can coagulate with the same rate $\Gamma_{11}^{10}=\Gamma_{11}^{01}$ and that when a particle is adjacent to a vacancy, a branching process can occur with rate $\Gamma_{10}^{11}=\Gamma_{01}^{11}$. In addition, when two adjacent sites are empty, a particle can spontaneously appear on a site (*input*) with (a finite) rate $\Gamma_{00}^{10}=\Gamma_{00}^{01}>0$. The dynamics of this RDCI model can be encoded by the following reactions:

$$\mathcal{A}\varnothing \leftrightarrow \varnothing\mathcal{A} \text{ with rate } \Gamma_{10}^{01}=\Gamma_{01}^{10}>0,$$

$$\varnothing\varnothing \rightarrow \varnothing\mathcal{A} \text{ and } \varnothing\varnothing\rightarrow\mathcal{A}\varnothing \text{ with rate } \Gamma_{00}^{01}=\Gamma_{00}^{10}>0,$$

$$\mathcal{A}\varnothing\rightarrow\mathcal{A}\mathcal{A} \text{ and } \varnothing\mathcal{A}\rightarrow\mathcal{A}\mathcal{A} \text{ with rate } \Gamma_{10}^{11}=\Gamma_{01}^{11},$$

$$\mathcal{A}\mathcal{A}\rightarrow\mathcal{A}\varnothing \text{ and } \mathcal{A}\mathcal{A}\rightarrow\varnothing\mathcal{A} \text{ with rate } \Gamma_{11}^{10}=\Gamma_{11}^{01}. \quad (47)$$

In order to apply the (conventional) IPDF method, we require that the coagulation and diffusion rates are [9–12]

$$\Gamma_{10}^{01}=\Gamma_{11}^{01} \text{ and } \Gamma_{01}^{10}=\Gamma_{11}^{10}. \quad (48)$$

Therefore with Eq. (48), we have $\Gamma_{10}^{01}=\Gamma_{11}^{01}=\Gamma_{10}^{01}=\Gamma_{11}^{10}>0$ and are left with $4-1=3$ independent reaction rates for the model (47).

Before proceeding with the solution of this model some comments are in order. To our knowledge this model has been studied in the continuum limit on an infinite chain (for the translationally invariant situation) by Doering and ben-Avraham [9]. The latter obtained, in this limit, the stationary concentration of particles, the stationary interparticle function, and the relaxation spectrum as the zeroes of some Airy functions. In their work, Doering and co-workers considered an infinite chain with lattice spacing $\triangle x$ (here $\triangle x=1$) in the continuum limit ($\triangle x\rightarrow 0$). On this infinite chain, the reactions occuring are the symmetric diffusion and coagulation with rates $\Gamma_{10}^{01}=\Gamma_{01}^{10}=\Gamma_{11}^{01}=\Gamma_{11}^{10}=\mathcal{D}/(\triangle x)^2$, the symmetric branching processes with rate $\Gamma_{10}^{11}=\Gamma_{01}^{11}=v/\triangle x$, and the input of particles with rate $\Gamma_{00}^{10}=\Gamma_{00}^{01}=R\triangle x$.

In this section we want to solve the model (47) with the restrictions (48) on a finite and periodic lattice of L sites. According to the definitions (10), for this section we have $\alpha_1=\alpha_2\equiv\alpha\equiv 2(C_1-A_1)=2\Gamma_{01}^{10}>0$, $\beta_1=\beta_2\equiv\beta\equiv -2D_1=2B_1=2(\Gamma_{10}^{01}+\Gamma_{10}^{11}-\Gamma_{00}^{10})$, $\delta=A_1+A_2=2\Gamma_{00}^{10}>0$, and $\gamma+\delta=2(2\Gamma_{01}^{10}+\Gamma_{10}^{11})$.

Hereafter, we thus solve the model (47) with the constraints (48), which is a model described by $4-1=3$ independent paramaters (reaction rates), for the case where the input of particles is nonvanishing (i.e., $\delta>0$) and $D\neq 0$. The cases where $\delta=0$ have been extensively studied in the continuum limit [9,10] as well as on discrete lattices [11]: these models have been mapped onto free-fermion models [11,2,6]. In addition, when $D=\beta=0$, for the translationally invariant situation and an initial density $\rho(0)$ of particles, we have [when $B\neq C$, see footnote (2)]

$$\langle n_x(t)\rangle = \frac{A}{C-B} + \left[\rho(0) - \frac{A}{C-B}\right] e^{-2(C-B)t}. \quad (49)$$

For the model under consideration here, we have $\mu \equiv \sqrt{\alpha/\beta}$ and $q=\sqrt{\alpha\beta}$. As $\alpha>0$ and $\beta \geqslant 0$, with $\beta=0 \Rightarrow D=0$, we focus on the case where $q \neq 0$. One should be cautious with the fact that q and μ can be imaginary, in which case the sign ambiguity is fixed by requiring that $q\mu=\alpha$.

The equation of motion of the string function associated with this model is thus of the form (9), which has been solved in Sec. IV.

To compute the relevant physical quantities, we proceed as in the previous section. In fact the expressions for the density, correlation function, and the current of particles can be immediately obtained from Eqs. (40)–(45) when $b/a=1$. Hereafter, for the sake of completness and clarity we quote these expressions

$$\langle n_x(\infty)\rangle = 1-\mu\left[\frac{J_{L+\gamma/\delta}(2q/\delta)Y_{1+\gamma/\delta}(2q/\delta)-Y_{L+\gamma/\delta}(2q/\delta)J_{1+\gamma/\delta}(2q/\delta)}{J_{L+\gamma/\delta}(2q/\delta)Y_{\gamma/\delta}(2q/\delta)-Y_{L+\gamma/\delta}(2q/\delta)J_{\gamma/\delta}(2q/\delta)}\right], \tag{50}$$

$$\langle n_x n_{x+1}\rangle(\infty) = 2\langle n_x(\infty)\rangle - 1 + \mu^2\left[\frac{J_{L+\gamma/\delta}(2q/\delta)Y_{2+\gamma/\delta}(2q/\delta)-Y_{L+\gamma/\delta}(2q/\delta)J_{2+\gamma/\delta}(2q/\delta)}{J_{L+\gamma/\delta}(2q/\delta)Y_{\gamma/\delta}(2q/\delta)-Y_{L+\gamma/\delta}(2q/\delta)J_{\gamma/\delta}(2q/\delta)}\right]. \tag{51}$$

We again note that the expressions (50) and (51) are independent of the site label x and therefore correspond to translationally invariant stationary quantities.

As for the BCBD model, from Eqs. (50) and (51) one can check that the RDCI model is a correlated system of interacting particles, characterized by a translationally invariant but correlated stationary distribution: $\langle n_x n_{x+1}\rangle(\infty) \neq \langle n_x(\infty)\rangle\langle n_{x+1}(\infty)\rangle = [\langle n_x(\infty)\rangle]^2$.

To study the dynamical properties of the model, we need the relaxation spectrum $\{E_i\}$, $i=1,\ldots,L$, of the string function. As established in the previous section, the latter is obtained as the set of zeros of the following Lommel function: $R_{L-1,1-E}(2q/\delta)=0$, where $E \equiv (q\lambda-\gamma)/\delta$, which are computed solving the associated eigenvalue problem (20). Notice that when q is real the matrix $\mathcal{M}$, Eq. (21) is Hermitian, otherwise (q is imaginary) $\mathcal{M}$ is anti-Hermitian.

Again the spectrum $\{E\}$ (and therefore $\{q\lambda\}$) turns out to be *real* (even when $\mathcal{M}$ is anti-Hermitian) and symmetrically distributed around $L/2$, which is an eigenvalue when L is even.

When $\Gamma_{00}^{10} > \Gamma_{01}^{10}+\Gamma_{01}^{11}$, then q is imaginary with $0 < |q|/\delta \leqslant 1/2$. When q is imaginary, the eigenvalues are not generally integers, but for the central part of the spectrum (when the eigenvalues are close to $L/2$), the eigenvalues approach integer values. This is not the case for the smallest eigenvalue $E^* = \min_E\{E\}$, which is not an integer and depends on the size of the system: $E^*=\epsilon_L>1$. However, for $L \gg 1$, $\epsilon_L \to \epsilon_\infty$ and E^* is a constant: $E^*=\epsilon_\infty>1$. The expression (22) can again be considered as an excellent approximation for systems of size $L \gg 1$ and, in particular, for ϵ_∞. We can then show that for q imaginary, the quantity $\epsilon_L > 1$. In fact one can compute (approximative) bounds for ϵ_L with $L \gg 1$. This is achieved with help of Eq. (22), namely, $1 < \epsilon_{L=6} \leqslant 3-\sqrt{2+\frac{1}{4}\sqrt{13}}$.

When $\Gamma_{00}^{10} < \Gamma_{01}^{10}+\Gamma_{01}^{11}$, q is real. The eigenvalues are still symetrically distributed around $L/2$ (which is still an eigenvalue when L is even). In this case, the smallest eigenvalue $E^* \equiv \epsilon_L$ can be negative (e.g., $\epsilon_{L=6}<0$ if $q/\delta > 1.299\,07\ldots$). However, $-2\Gamma_{01}^{10}/\Gamma_{00}^{10} \lesssim \epsilon_L \lesssim 1$ (in fact, $-2\Gamma_{01}^{10}/\Gamma_{00}^{10} < \epsilon_{L=6} \leqslant 1$), and thus, when q is real, we have, $q\lambda^* = \epsilon_L\delta+\gamma = 2[2\Gamma_{01}^{10}+\Gamma_{10}^{11}+(\epsilon_L-1)\Gamma_{00}^{10}] \geqslant 2(\Gamma_{01}^{10}+\epsilon_L\Gamma_{00}^{10}) > 0$. Again, for large systems ($L \gg 1$) the expression of ϵ_L is well approximated by the exact expression (22) of $\epsilon_{L=6}$.

When $\Gamma_{00}^{10}=\Gamma_{01}^{10}+\Gamma_{01}^{11}$, then $D=q=0$ and we recover Eq. (49), with $B=D=0$.

In definitive, the long-time dynamics of large systems is governed by the eigenvalue $E^*>0$ according to Eq. (22) for $L \gg 1$. This means that the inverse of the relaxation time of the system reads

$$q\lambda^* = E^*\delta+\gamma = \epsilon_L\delta+\gamma = 2[2\Gamma_{01}^{10}+\Gamma_{10}^{11}+(\epsilon_L-1)\Gamma_{00}^{10}] > 0. \tag{52}$$

The dynamical parts of the above quantities are obtained similarly from Eqs. (43) and (44), setting $b/a=1$ in these expressions,

$$\langle n_x(t)\rangle - \langle n_x(\infty)\rangle = \mu\sum_{E_i} \mathcal{A}_{E_i} e^{-(E_i\delta+\gamma)t}[J_{1-E_i}(2q/\delta)Y_{L-E_i}(2q/\delta) - Y_{1-E_i}(2q/\delta)J_{L-E_i}(2q/\delta)], \tag{53}$$

$$\langle n_x n_{x+1}\rangle(t) - \langle n_x n_{x+1}\rangle(\infty) = \{\langle n_x(t)\rangle + \langle n_{x+1}(t)\rangle - 2\langle n_x(\infty)\rangle\} + \mu^2\sum_{E_i} \mathcal{A}_{E_i} e^{-(E_i\delta+\gamma)t}[J_{2-E_i}(2q/\delta)Y_{L-E_i}(2q/\delta) - Y_{2-E_i}(2q/\delta)J_{L-E_i}(2q/\delta)], \tag{54}$$

where the coefficients $\mathcal{A}_{E_i}$ have been computed in the translationally invariant situation in Eqs. (25)–(30).

From Eqs. (53), (54), and (24), we can compute for this RDCI model the approximative expression for all two-point correlation functions according to the scheme (33)–(36).

The long-time behavior of the quantities (53) and (54) are also obtained as explained in the previous section and in particular the long-time behavior of the density is given by Eq. (45), where the smallest eigenvalue E^* is the quantity obtained in Eq. (52).

As in the case of the model BCBD, and for the same reasons, the density and the two-point correlation functions (instantaneous and noninstantaneous) relax exponentially fast to the steady state with an inverse of the relaxation time given by Eq. (52).

Another relevant quantity that one can compute for this model is the so-called interparticle function $p_{x,y}(t)$ [9,10]. The latter gives the probability, a time t, for a particle at site $x-1$ to have as a next neighbor, a particle at a distant site $y-x>0$. To obtain $p_{x,y}(t)$, we set $a=b=1$ and the string function $S_{x,y}(t)$ reduces to the empty-interval function [9,10] $S_{x,y}(t)=\langle\Pi_{j=x}^{y-1}(1-n_j)\rangle(t)$, which is associated with the probability of having a sequence of holes starting from the site x and of length $y-x$. As shown by Doering and coworkers [9,10], it is possible to relate the density of particles, the empty-interval function $S_{x,y}$, and the interparticle function $p_{x,y}(t)$: $S_{x,y+1}(t)-2S_{x,y}(t)+S_{x,y-1}(t)=\langle(1-n_x)\cdots(1-n_{y-2})[n_{y-1}-(1-n_{y-1})n_y]\rangle(t)$, Therefore, with $\rho(t)=(1/L)\Sigma_j\langle n_j(t)\rangle$, for the translationally invariant situation [the inverse of $\rho(t)$ measures the average distance between adjacent particles], we have

$$p_{x,y}(t)=p_{y-x}(t)=\frac{[S_{x,y+1}(t)-2S_{x,y}(t)+S_{x,y-1}(t)]}{\rho(t)}. \tag{55}$$

In particular, in the stationary case, we have

$$\begin{aligned}p_{x,y}(\infty)&=\frac{S_{x,y+1}(\infty)-2S_{x,y}(\infty)+S_{x,y-1}(\infty)}{\rho(\infty)}\\&=\mu^{y-x}\left\{\frac{J_{L+\gamma/\delta}\left(\frac{2q}{\delta}\right)\left[\mu Y_{y-x+1+\gamma/\delta}\left(\frac{2q}{\delta}\right)+\mu^{-1}Y_{y-x-1+\gamma/\delta}\left(\frac{2q}{\delta}\right)-2Y_{y-x+\gamma/\delta}\left(\frac{2q}{\delta}\right)\right]}{J_{L+\gamma/\delta}\left(\frac{2q}{\delta}\right)\left[Y_{\gamma/\delta}\left(\frac{2q}{\delta}\right)-\mu Y_{1+\gamma/\delta}\left(\frac{2q}{\delta}\right)\right]-Y_{L+\gamma/\delta}\left(\frac{2q}{\delta}\right)\left[J_{\gamma/\delta}\left(\frac{2q}{\delta}\right)-\mu J_{1+\gamma/\delta}\left(\frac{2q}{\delta}\right)\right]}\right\}\\&\quad-\mu^{y-x}\left\{\frac{Y_{L+\gamma/\delta}\left(\frac{2q}{\delta}\right)\left[\mu J_{y-x+1+\gamma/\delta}\left(\frac{2q}{\delta}\right)+\mu^{-1}J_{y-x-1+\gamma/\delta}\left(\frac{2q}{\delta}\right)-2J_{y-x+\gamma/\delta}\left(\frac{2q}{\delta}\right)\right]}{J_{L+\gamma/\delta}\left(\frac{2q}{\delta}\right)\left[Y_{\gamma/\delta}\left(\frac{2q}{\delta}\right)-\mu Y_{1+\gamma/\delta}\left(\frac{2q}{\delta}\right)\right]-Y_{L+\gamma/\delta}\left(\frac{2q}{\delta}\right)\left[J_{\gamma/\delta}\left(\frac{2q}{\delta}\right)-\mu J_{1+\gamma/\delta}\left(\frac{2q}{\delta}\right)\right]}\right\}.\end{aligned} \tag{56}$$

To conclude this section, let us point out the fact that the results (50)–(56) can be immediately generalized to systems which, in addition to the processes (47), also include the (adjacent) pair-creation reaction $\varnothing\varnothing\rightarrow\mathcal{A}\mathcal{A}$, with rate Γ_{00}^{11}. In fact, it suffices to replace Γ_{00}^{10} with $\Gamma_{00}^{10}+\Gamma_{00}^{11}$ in Eqs. (50)–(56).

We also would like to emphasize the fact that the expressions (50)–(56) are different from those obtained by Doering and ben-Avraham [9] who considered the continuum limit of this model on an infinite chain.

VII. SOLUTION OF MODELS (WHICH CANNOT BE MAPPED ONTO FREE-FERMION SYSTEMS) VIA SIMILARITY TRANSFORMATIONS

In Secs. V and VI we have solved two different reaction-diffusion (the BCBD and the RDCI) models, which cannot be mapped onto free-fermion systems. It is therefore natural to wonder whether or not there exists a mapping between these models, transforming the empty-interval function $\langle\Pi_{j=x}^{y-1}(1-n_j)\rangle(t)$ into another string function of the form $\langle\Pi_{j=x}^{y-1}(1-[b/a]n_j)\rangle(t)$. In this section we study the class of models that can be obtained from the (RDCI) model, analyzed in Sec. VI through a class of local similarity transformations. In so doing we will answer the following questions.

(i) Does a similarity transformation, which maps the RDCI onto the BCBD model and the empty-interval function (with $a=b$) onto a string function (with $b/a\neq 1$) exist?

(ii) If so, does the mapping provide the solution of the model BCBD for the same constraints (39) considered in Sec. V?

Let us consider the *original* stochastic Hamiltonian H, through the (local) similarity transformation $\mathcal{B}$; define $\tilde{H}$ as [11,2,13]

$$\tilde{H}\equiv\mathcal{B}^{-1}H\mathcal{B}. \tag{57}$$

Because of the requirement that $\langle\tilde{O}(t)\rangle[\tilde{H},|\tilde{P}(0)\rangle]=\langle O(t)\rangle[H,|P(0)\rangle]$, which implies that

$$\langle\tilde{\chi}|Oe^{-Ht}|P(0)\rangle=\langle\tilde{\chi}|\tilde{O}e^{-\tilde{H}t}|\tilde{P}(0)\rangle$$
$$=\langle\tilde{\chi}|\tilde{O}\mathcal{B}^{-1}e^{-Ht}\mathcal{B}|\tilde{P}(0)\rangle$$
$$=\langle\tilde{\chi}|O\mathcal{B}\mathcal{B}^{-1}e^{-Ht}\mathcal{B}\mathcal{B}^{-1}|P(0)\rangle,$$

it is clear that under this transformation, the observable O, and the initial state $|P(0)\rangle$ transform according to $\tilde{O}\equiv O\mathcal{B}$ and $|\tilde{P}(0)\rangle\equiv\mathcal{B}^{-1}|P(0)\rangle$, where we assume homogeneous (uncorrelated, yet random) initial states with density $0\leqslant\rho(0)\leqslant 1$ of particles,

$$|P(0)\rangle=\begin{pmatrix}1-\rho(0)\\ \rho(0)\end{pmatrix}.$$

In this section we focus on local transformations of the form [2,11,13]

$$\mathcal{B}=\otimes_{j=1}^{L}\mathbf{b}_j, \tag{58}$$

where $\mathbf{b}_j$ denotes a 2×2 matrix acting at site j such that the stochasticity condition $\langle\tilde{\chi}|\mathcal{B}\tilde{H}=0$ is fulfilled. In addition, in order to transform the empty-interval function into a more general string function we want to consider transformations that map the operator $1-n_j$ onto $\widetilde{1-n_j}\equiv 1-rn_j$, where $r\equiv b/a$. Peschel *et al.* [12] have shown that the transformation (58), with

$$\mathbf{b}_j\equiv\begin{pmatrix}1 & 1-r\\ 0 & r\end{pmatrix}_j \tag{59}$$

satisfies this property. Through this transformation, according to Eq. (57), the stochastic Hamiltonian related to the RDCI model transforms into the following stochastic Hamiltonian:

$$\tilde{H}_{j,j+1}=\begin{pmatrix}\tilde{\Gamma}_{00}^{00} & \tilde{\Gamma}_{01}^{00} & \tilde{\Gamma}_{10}^{00} & \tilde{\Gamma}_{11}^{00}\\ \tilde{\Gamma}_{00}^{01} & \tilde{\Gamma}_{01}^{01} & \tilde{\Gamma}_{10}^{01} & \tilde{\Gamma}_{11}^{01}\\ \tilde{\Gamma}_{00}^{10} & \tilde{\Gamma}_{01}^{10} & \tilde{\Gamma}_{10}^{10} & \tilde{\Gamma}_{11}^{10}\\ \tilde{\Gamma}_{00}^{11} & \tilde{\Gamma}_{01}^{11} & \tilde{\Gamma}_{10}^{11} & \tilde{\Gamma}_{11}^{11}\end{pmatrix},$$

where the nondiagonal entries read

$$\tilde{\Gamma}_{00}^{01}=\tilde{\Gamma}_{00}^{10}/r=\Gamma_{00}^{10}/r,\quad \tilde{\Gamma}_{00}^{11}=0,$$
$$\tilde{\Gamma}_{01}^{00}=\tilde{\Gamma}_{10}^{00}=(2\Gamma_{00}^{10}-\Gamma_{10}^{11})(r-1)/r,$$
$$\tilde{\Gamma}_{01}^{10}=\tilde{\Gamma}_{10}^{01}=[(\Gamma_{10}^{11}-\Gamma_{00}^{10})(r-1)+r\Gamma_{01}^{10}]/r,$$
$$\tilde{\Gamma}_{01}^{11}=\tilde{\Gamma}_{10}^{11}=\Gamma_{10}^{11}/r,$$
$$\tilde{\Gamma}_{11}^{00}=2(r-1)[(r-1)(\Gamma_{10}^{11}-\Gamma_{00}^{10})+r\Gamma_{01}^{10}]/r,$$
$$\tilde{\Gamma}_{11}^{01}=\tilde{\Gamma}_{11}^{10}=[\Gamma_{00}^{10}(r-1)^2+(2-r)((r-1)\Gamma_{10}^{11}+r\Gamma_{01}^{10})]/r. \tag{60}$$

The uncorrelated and homogeneous, but random, initial state becomes

$$|\tilde{P}(0)\rangle=\begin{pmatrix}1-\dfrac{\rho(0)}{r}\\ \dfrac{\rho(0)}{r}.\end{pmatrix}. \tag{61}$$

One has to ensure that all the reaction rates appearing in Eq. (60) are $\geqslant 0$, which requires that $r\geqslant 0$ and that $0\leqslant\rho(0)/r\leqslant 1$. Therefore we have the necessary condition $r\geqslant\rho(0)\geqslant 0$.

According to the transformations (58) and (59), the empty-interval function $\langle\Pi_{j=x}^{y-1}(1-n_j)\rangle(t)$ is mapped onto the string function $\langle\Pi_{j=x}^{y-1}(\widetilde{1-n_j})\rangle(t)=\langle\Pi_{j=x}^{y-1}(1-rn_j)\rangle(t)$ and therefore, the statistical quantities for the model described by the stochastic Hamiltonian $\tilde{H}$ are obtained from the related quantities computed in Sec. VI for the model RDCI as

$$\langle n_x(t)\rangle_{\tilde{H},|\tilde{P}(0)\rangle}=\frac{1}{r}\langle n_x(t)\rangle_{H,|P(0)\rangle};$$

$$\langle n_x n_{x+y}\rangle_{\tilde{H},|\tilde{P}(0)\rangle}(t)=\frac{1}{r^2}\langle n_x n_{x+y}\rangle_{H,|P(0)\rangle}(t)\quad (y>0), \tag{62}$$

where $\langle n_x(t)\rangle_{H,|P(0)\rangle}$ and $\langle n_x n_{x+1}\rangle_{H,|P(0)\rangle}(t)$ have been computed in Eqs. (53) and (54).

We will now consider two specific models described by Eq. (60).

(a) To answer the questions (i) and (ii), we seek, through the mapping (58) and (59), a model of the BCBD type and thus require $\tilde{\Gamma}_{10}^{01}=\tilde{\Gamma}_{01}^{10}=\tilde{\Gamma}_{11}^{00}=0$, as for the BCBD model considered in Sec. V and infer from Eq. (60), $(\Gamma_{10}^{11}-\Gamma_{00}^{10})(r-1)+r\Gamma_{01}^{10}=0$ $(r\neq 1)$, which implies

$$r=\frac{\Gamma_{10}^{11}-\Gamma_{00}^{10}}{\Gamma_{10}^{11}+\Gamma_{01}^{10}-\Gamma_{00}^{10}}\geqslant 0. \tag{63}$$

Replacing the expression (63) in Eq. (60), we obtain the reaction rates of the BCBD model in terms of the rates of the original (RDCI) model. In order to have a physical BCBD model, we have to require the reaction rates $\tilde{\Gamma}$'s to be positive. We now take advantage from the fact that a version of the BCBD model has been solved in Sec. V, where $\tilde{\Gamma}_{10}^{00}=(\tilde{\Gamma}_{11}^{10}/\tilde{\Gamma}_{00}^{10})(2\tilde{\Gamma}_{00}^{10}-\tilde{\Gamma}_{10}^{11})$. It is therefore possible to check, from Eqs. (60) and (63), that this relation still holds in this case and parametrize the reaction rates (60) as follows:

$$\tilde{\Gamma}^{10}_{00}=\Gamma^{10}_{00}/r, \quad \tilde{\Gamma}^{11}_{10}=\Gamma^{11}_{10}/r, \quad \tilde{\Gamma}^{10}_{11}=(r-1)\tilde{\Gamma}^{10}_{00},$$

$$\tilde{\Gamma}^{00}_{10}=\frac{\tilde{\Gamma}^{10}_{11}}{\tilde{\Gamma}^{10}_{00}}(2\tilde{\Gamma}^{10}_{00}-\tilde{\Gamma}^{11}_{10}), \quad r=1+\frac{\tilde{\Gamma}^{10}_{11}}{\tilde{\Gamma}^{10}_{00}}>1. \tag{64}$$

The requirement of positivity of these rates (64) leads to $2\tilde{\Gamma}^{10}_{00}>\tilde{\Gamma}^{10}_{11}$. Thus, the reaction rates (64) describing the BCBD model from the RDCI model are identical to those considered in Sec. V for solving the BCBD directly from the generalized string function. It is therefore easy to obtain the density and the correlation functions of the BCBD model from the RDCI model inverting the relation (64) and using Eqs. (61) and (62). As an illustration, we consider the BCBD model with rates $\tilde{\Gamma}^{10}_{00}=1$, $\tilde{\Gamma}^{11}_{10}=1/2$, $\tilde{\Gamma}^{10}_{11}=2$, and $\tilde{\Gamma}^{00}_{10}=3$, which imply that $r=3$. Such a model is thus mapped onto the RDCI model with the rates $\Gamma^{10}_{00}=3, \Gamma^{11}_{10}=3/2, \Gamma^{10}_{01}=\Gamma^{10}_{11}=1$. In this case the density of the BCBD model is related to the density of the RDCI model according to Eq. (62),

$$\langle n_x(t)\rangle^{BCBD}_{(\tilde{\Gamma}^{10}_{00}=1,\tilde{\Gamma}^{11}_{10}=1/2,\tilde{\Gamma}^{10}_{11}=2,\tilde{\Gamma}^{00}_{10}=3;|\tilde{P}(0)\rangle)}$$

$$=\frac{1}{3}\langle n_x(t)\rangle^{RDCI}_{(\Gamma^{10}_{00}=3,\Gamma^{11}_{10}=3/2,\Gamma^{10}_{01}=\Gamma^{10}_{11}=1;|P(0)\rangle)}.$$

In particular we have seen that the stationary density is independent of the initial state and thus, in this case, we have [see Eqs. (40) and (50)]

$$\langle n_x(\infty)\rangle^{BCBD}_{(\tilde{\Gamma}^{10}_{00}=1,\tilde{\Gamma}^{11}_{10}=1/2,\tilde{\Gamma}^{10}_{11}=2,\tilde{\Gamma}^{00}_{10}=3)}=\frac{1}{3}\langle n_x(\infty)\rangle^{RDCI}_{(\Gamma^{10}_{00}=3,\Gamma^{11}_{10}=3/2,\Gamma^{10}_{01}=\Gamma^{10}_{11}=1)}$$

$$=\frac{1}{3}+\frac{i\sqrt{2}}{3}\left[\frac{J_{L+1/6}(\sqrt{2}i/3)Y_{7/6}(\sqrt{2}i/3)-Y_{L+1/6}(\sqrt{2}i/3)J_{7/6}(\sqrt{2}i/3)}{J_{L+1/6}(\sqrt{2}i/3)Y_{1/6}(\sqrt{2}i/3)-Y_{L+1/6}(\sqrt{2}i/3)J_{1/6}(\sqrt{2}i/3)}\right]$$

$$\simeq 0.2401 \quad (L\gg 1).$$

We are now in a position to answer the questions (i) and (ii).

We have shown that there exists a similarity transformation (58) and (59) that transforms the empty-interval function onto the generalized string function and that maps the RDCI onto the BCBD model, with the same constraints of solvability as the constraints (39) imposed in Sec. V. We conclude that the present approach and the method devised in Sec. V are equivalent.

One additional comment on this equivalence is however useful at this point. Although both mentioned methods are equivalent, the method devised in Sec. V is in a sense more convenient because it is direct: solving the equation of motion of the adequate (generalized) string function solves directly the BCBD model. Conversely, via the similarity transformation we first solve the RDCI model, which is a task of the same difficulty as that of solving the BCBD model, and then find an adequate and nontrivial similarity transformation (where the new reaction rates should be interpreted correctly in terms of the original ones).

(b) Let us now consider a model that can be solved from the solution of the RDCI model via the similarity transformation (58) and (59). The model under consideration is a reversible diffusion coagulation with particles input and pair annihilation (RDCIPA), in which the dynamics can be symbolized by the reactions

$$\mathcal{A}\varnothing \leftrightarrow \varnothing\mathcal{A} \text{ with rate } \tilde{\Gamma}^{01}_{10}=\tilde{\Gamma}^{10}_{01}>0,$$

$$\varnothing\varnothing\rightarrow\varnothing\mathcal{A}, \text{ and } \varnothing\varnothing\rightarrow\mathcal{A}\varnothing \text{ with rate } \tilde{\Gamma}^{01}_{00}=\tilde{\Gamma}^{10}_{00}>0,$$

$$\mathcal{A}\varnothing\rightarrow\mathcal{A}\mathcal{A} \text{ and } \varnothing\mathcal{A}\rightarrow\mathcal{A}\mathcal{A} \text{ with rate } \tilde{\Gamma}^{11}_{10}=\tilde{\Gamma}^{11}_{01},$$

$$\mathcal{A}\mathcal{A}\rightarrow\mathcal{A}\varnothing \text{ and } \mathcal{A}\mathcal{A}\rightarrow\varnothing\mathcal{A} \text{ with rate } \tilde{\Gamma}^{10}_{11}=\tilde{\Gamma}^{01}_{11},$$

$$\mathcal{A}\mathcal{A}\rightarrow\varnothing\varnothing \text{ with rate } \tilde{\Gamma}^{00}_{11}. \tag{65}$$

This model (RDCIPA) can be obtained from the RDCI model via the similarity transformation (58) and (59). Imposing $\Gamma^{11}_{10}=2\Gamma^{10}_{00}$ in Eq. (60) we get the following reaction rates:

$$\tilde{\Gamma}^{10}_{00}=\tilde{\Gamma}^{01}_{00}=\Gamma^{10}_{00}/r, \quad \tilde{\Gamma}^{11}_{10}=\tilde{\Gamma}^{11}_{01}=2\Gamma^{10}_{00},$$

$$\tilde{\Gamma}^{00}_{11}=2(r-1)\tilde{\Gamma}^{10}_{01},$$

$$\tilde{\Gamma}^{10}_{11}=\tilde{\Gamma}^{01}_{11}=(2-r)\tilde{\Gamma}^{10}_{01}+(r-1)\tilde{\Gamma}^{10}_{00},$$

$$\tilde{\Gamma}^{10}_{01}=\tilde{\Gamma}^{10}_{01}=\frac{r-1}{2}\tilde{\Gamma}^{11}_{10}+\Gamma^{01}_{10}. \tag{66}$$

For this RDCIPA model, we have three (positive) independent parameters $r\geqslant 1$, $\Gamma^{10}_{00}\geqslant 0$, and $\Gamma^{10}_{01}\geqslant 0$. The positivity of the reaction rates (65) and the physical meaning of the initial state requires the following constraints:

$$\tilde{\Gamma}^{01}_{11}=(2-r)\tilde{\Gamma}^{10}_{01}+(r-1)\tilde{\Gamma}^{10}_{00}\geqslant 0, \quad r\geqslant 1, \quad 0\leqslant\rho(0)\leqslant r. \tag{67}$$

Thus, for the model RDCIPA (65) described by the reaction rates (66) with the restrictions (67), the density and the correlation functions can be computed from the results of the model RDCI according to Eq. (62), for homogeneous (but random) initial states described by Eq. (61).

VIII. PROPAGATION OF A WAVE FRONT AND THE FISHER WAVES

At the end of Sec. II, we have stated that some reaction-diffusion models are described at the mean-field level and in the continuum limit by nonlinear partial differential equation of the Fisher type (5) [3]. In this section we show that for some choices of the parameters (reaction rates) the mean-field formulation of the models BCBD and RDCI gives rise to Fisher-type equations. Then, from the results obtained in Secs. V and VI, we study the propagation of the wave front from a microscopic point of view (in so doing, the correlation between particles is taken into account exactly). We show that the scenario predicted by Fisher's theory fails in one spatial dimension for the models under consideration. In this whole section, we adopt the same notation as that introduced at the end of the Sec. II.

(i) For the BCBD model, setting

$$\phi_{BCBD}\equiv\frac{(2\Gamma_{00}^{10}+\Gamma_{10}^{00}-\Gamma_{10}^{11})+\sqrt{(\Gamma_{10}^{11})^2+(\Gamma_{10}^{00})^2+4\Gamma_{00}^{10}\Gamma_{11}^{10}-2\Gamma_{10}^{11}\Gamma_{10}^{00}}}{2[(\Gamma_{10}^{00}+\Gamma_{00}^{10})-(\Gamma_{10}^{11}+\Gamma_{11}^{10})]}, \tag{68}$$

and with the additional definitions, we have

$$k_1^{BCBD}\equiv 2\sqrt{(\Gamma_{10}^{11})^2+(\Gamma_{10}^{00})^2+4\Gamma_{00}^{10}\Gamma_{11}^{10}-2\Gamma_{10}^{11}\Gamma_{10}^{00}}>0 \quad \text{and} \quad k_2^{BCBD}\equiv 2[\Gamma_{10}^{11}+\Gamma_{11}^{10}-(\Gamma_{10}^{00}+\Gamma_{00}^{10})]>0, \tag{69}$$

where the reaction rates appearing in Eqs. (68) and (69) are those defined in Eq. (39).

(ii) For the RDCI model, setting

$$\phi_{RDCI}\equiv\frac{2\Gamma_{00}^{10}-\Gamma_{10}^{11}+\sqrt{(\Gamma_{10}^{11})^2+4\Gamma_{00}^{10}\Gamma_{01}^{10}}}{2[\Gamma_{00}^{10}-(\Gamma_{10}^{01}+\Gamma_{10}^{11})]} \tag{70}$$

and with the additional definitions, we have

$$k_1^{RDCI}\equiv 2\sqrt{(\Gamma_{10}^{11})^2+4\Gamma_{00}^{10}\Gamma_{01}^{10}}>0 \quad \text{and} \quad k_2^{RDCI}\equiv 2[(\Gamma_{01}^{10}+\Gamma_{10}^{11})-\Gamma_{00}^{10}]>0. \tag{71}$$

The reaction rates appearing in Eqs. (70) and (71) are those introduced in Sec. VI.

Under the conditions (i) and (ii), at the continuum mean field level, we have for the BCBD and RDCI models with $\tilde{\rho}_{MF}^{l}(x,t)\equiv\rho_{MF}^{l}(x,t)-\phi_l$, where $l=$BCBD and RDCI, equations of motion that are Fisher's equations:

$$\frac{\partial}{\partial t}\tilde{\rho}_{MF}^{l}(x,t)=\frac{k_2^l}{2}\frac{\partial^2}{\partial x^2}\tilde{\rho}_{MF}^{l}(x,t)+k_1^l\tilde{\rho}_{MF}^{l}(x,t)-k_2^l[\tilde{\rho}_{MF}^{l}(x,t)]^2. \tag{72}$$

Assuming L to be even and relabeling the sites of the chain according to the shift: $x\to x-L/2$, we consider an initial inhomogeneous configuration with

$$\langle n_x(0)\rangle=\begin{cases}\langle n_x(\infty)\rangle & \text{if } x\in[-L/2,0]\\ 0 & \text{otherwise.}\end{cases} \tag{73}$$

We want now to compare the prediction of the mean-field theory with the results obtained directly from the microscopic results derived in Secs. V and VI and thus compute the time-dependent position $X(t)$ of the wave front and its time-dependent width $w(t)$. This is done according to the formulas [14]

$$X(t)=\sum_{x=-L/2}^{L/2}\frac{\langle n_x(t)\rangle}{\langle n_x(\infty)\rangle}-\frac{L}{2} \quad \text{and}$$

$$w(t)^2=2\sum_{x=-L/2}^{L/2}\frac{x\langle n_x(t)\rangle}{\langle n_x(\infty)\rangle}-\left(\frac{L}{2}\right)^2-X(t)^2. \tag{74}$$

With the help of Eqs. (42) and (52), and denoting with $q_l,\delta_l,\gamma_l,E_l^*$ the quantities related to the model $l\in$(BCBD,RDCI) and defined (computed) in Secs. V and VI, we obtain in the long-time regime $(E_l^*\delta_l+\gamma_l)t\gg 1$ where the time scales as $t\propto L^2$,

$$X_l(t)=\sqrt{2u_l(E_l^*\delta_l+\gamma_l)t}[1+O(e^{-(1/2)\sqrt{L^2/u_l}})] \tag{75}$$

and

$$w_l(t)=\sqrt{u_l(E_l^*\delta_l+\gamma_l)t}[1+O(e^{-(1/2)\sqrt{L^2/u_l}})], \tag{76}$$

where we have introduced the parameter $u_l\equiv L^2/2(E_l^*\delta_l+\gamma_l)t=O(1)$.

From these exact results, it appears that the location of the wave front moves as $\sqrt{t}$. Moreover, in contrast to Fisher's mean-field theory, the width of the wave front broadens as $\sqrt{t}$. These results, which have also been observed in the continuum limit for the one-dimensional reversible diffusion-coagulation (without input) model [14], confirm that in one spatial dimension the mean-field Fisher's picture fails. In fact Riordan *et al.* [14] have argued on the basis of extensive numerical computations for the reversible diffusion-coagulation (without input) model that in higher dimension

($\geqslant 4$) the latter model is in agreement with Fisher's mean-field predictions that the width of the wave front does not broaden. Recently other authors who studied the same model as Riordan *et al.* (in dimensions $d>1$) came to completely different conclusions [22].

Furthermore, for $\tilde{\rho}^l_{MF}$, Fisher's equation admits two (homogeneous) stationary states, namely, $\tilde{\rho}^l_{MF}(\infty)=k^l_1/k^l_2$, which is linearly stable [15] and $\tilde{\rho}^l_{MF}(\infty)=0$, which is linearly unstable [15]. This implies that at mean-field level, we would have for the stationary density, $\rho^l_{MF}(\infty)=\tilde{\rho}^l_{MF}(\infty)+\phi_l$, which corresponds to two possible steady states: $\rho^l_{MF}(\infty)=k^l_1/k^l_2+\phi_l$, $\rho^l_{MF}(\infty)=\phi_l$. However, from the exact expressions of the stationary density (40) and (50), we know that the models under consideration admit unique steady states that do not coincide with the mean-field prediction.

IX. SUMMARY AND CONCLUSION

In this work we have extended the conventional IPDF method. We introduced a string function, which is a natural generalization of the empty-interval function employed in the IPDF method. We derived the (five) constraints for the equations of motion to close [(see Eqs. (8) and (9)]. We solved the equation of motion of this string function on a periodic and finite lattice for the general form of a class of models that cannot be mapped onto free-fermion systems and that so far (to our knowledge) have been poorly understood [see Eq. (24)]. Then we specifically studied two models: The first one, which is a model with branching, coagulation, birth, and death processes (the BCBD model), can be viewed as a generalization of the voter model and/or as an epidemic model. The BCBD model is an example of a model that *cannot* be solved directly by the traditional IPDF method. For this model, under certain restrictions on the reaction rates [see Eq. (39)], the density, the noninstantaneous two-point, as well as the exact nearest neighbor (instantaneous) correlation functions have been analyzed: the steady states [see Eqs. (40) and (41)] as well as the dynamical approach towards the latter have been computed exactly [see Eqs. (43) and (44)]. In particular the relaxational spectrum as well as the the inverse of the relaxation time have been obtained [see Eq. (42)]. A similar analysis has been performed for a reversible diffusion-coagulation model with input of particles model. The latter (with the usual restriction that the coagulation rate coincides with the diffusion one) can be studied with help of the traditional IPDF method (the string function then reduces to the empty-interval function). In addition to the above-mentioned quantities, which we were able to compute also for the RDCI model, we calculated the stationary interparticle function [see Eq. (56)].

On the basis of the exact results, we have developed an approximative recursive scheme that allows to compute the (other) instantaneous two-point correlation functions [see Eq. (36)].

Studying these models, we observed that the latter is characterized by a translationally invariant stationary distribution for which, contrary to what happens to free-fermion systems, correlations are present: $\langle n_x n_{x+1}\rangle(\infty)\neq[\langle n_x(\infty)\rangle]^2$.

Later we studied the solution of the RDCI model and its implications on other systems related via similarity transformations. In particular, we considered a class of similarity transformations [see Eqs. (58) and (59)] that transforms the conventional empty-interval function into a more general string function. In so doing we saw that it is possible to map the RDCI model onto the BCBD one , which turns out to be solvable [via the similarity transformation (58) and (59)] with the same constraints encountered in Sec. V. We therefore conclude that the approaches of Secs. V and VII for solving the BCBD model are equivalent. However, it has to be noticed that working with the generalized string function as in Sec. V gives naturally access to the solution of the BCBD model without requiring the solution of another (RDCI) model.

We also have identified a model of reversible diffusion coagulation with particles input and pair annihilation (RDCIPA), which can be mapped, for some choices of the reaction rates [see Eqs. (66) and (67)], onto the RDCI model. For this RDCIPA model all the quantities previously computed for the RDCI can be immediately obtained via the similarity transformation [see Eq. (62)].

Finally we observed that on some parameter manifold, the mean-field approximation of the BCBD and RDCI models are described (in the continuum limit) by the so-called Fisher equations, which predict that an inhomogeneous initial configuration will evolve without broadening of the wave front in the density of particles. Computing the width of the wave front, which broadens as $\sqrt{t}$ [see Eq. (76)], we show that the Fisher's mean-field description fails in one dimension. Another failure of the mean-field theory is observed when one compares the mean-field predictions for the steady states of the density with the exact results.

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